Nonlinear Control Design

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Week 2 : Lecture 9 : Stability- Part 2

So, anyway so one of the things we I would I always like to emphasize is that the notion of uniform boundedness, uniform boundedness the way it is defined here does not imply stability or stability does not imply this in either direction ok. We will talk about the linear system matter later ok. Let us not get into that now because I want to move forward a little bit ok. Alright uniform stability the only difference see now that you understood stability hopefully well enough the other notions are relatively easier to follow actually yeah. So uniform stability all you do is you change this part. The initial dependence the initial time dependence of delta goes away that is what is uniform stability.

So whenever in all these stability notions we talk about uniformity it is somehow uniformity with respect to time ok. So here all that happens is the initial time dependence vanishes that is all and you can of course have so it is obviously a stronger property yeah it is a stronger property ok and we say that the equilibrium is unstable if it is neither stable nor uniformly stable ok if you have neither of these properties then it is unstable ok. So the first exercise that I mentioned here is of a Van der Pol oscillator right one of the most popular non-linear systems it is been used to model you know bio rhythms, heart beats and so on how the rhythmic how a lot of rhythmic things work why because the system exhibits something like a what is called limit cycle behavior yeah which means that the trajectory sort of you know get into this cyclic behavior in state space ok. So that is the same state is achieved after a certain time some kind of a periodicity yeah just like you have bio rhythms, heart beat and so on.

So Van der Pol oscillator is a very popular system for non-linear dynamics studies yeah what I want you to do is so this is the dynamics of the system this is the actual equation of the system what I will like you to do is tell me something about stability uniform stability or instability of the origin ok. You don't need to do any analysis I just want you to draw plot it in a computer for different initial conditions make the trajectories of the system in state space yeah what is the state space mean in this case there are two states x and x dot correct two dimensional system so two states x and x dot. So I want you to plot x and x dot x versus x dot for different initial conditions for different mu and just look at the plots and tell me whether it is stable or not because all you need to do is draw epsilon ball and delta ball right for any epsilon ball can you get a delta ball ok. So all I want you to do is make plots for different initial conditions x x dot ok. So start here and do see what happens yeah and for different values of mu you can choose some range of mu ok.

And then you I just want you to comment if origin is so I will say specifically yeah notice that origin is an equilibrium by the way origin means x equal to zero x dot equal to zero yeah it's an equilibrium yeah because if x is zero and x dot is zero then x double dot is zero it's an equilibrium valid equilibrium. So I just want you to comment by looking at the pictures no analysis nothing is required analysis is hard because without solving it how will you analyze alright ok. Now we go to a very fun example ok this is a example from Vidya Sagar's book very very nicely illustrates the notions of stability and uniform stability ok of course is not example is not by Vidya Sagar it is in Vidya Sagar's book but it is by some other control applied mathematician and very very nice ok. Because why is it nice because it is easy to solve so you can actually look at the system solutions and get a very good feel for whether it is stable unstable and all that ok. What is the system this very easy x dot is six t sin t minus twice t times x ok.

You know that this is very easy to solve right because all I have to do is take this here and integrate this right hand side with respect to time yeah and that's it I mean so the solutions actually turn out to be this guy yeah not difficult to see that this is the solution because if I take derivative of this yeah you will get six t sin t and the derivative of this is two t correct and then this is just the corresponding initial time terms exact same terms with opposite sign corresponding to initial time it is just the definite integral everybody is clear that this solution is sort of evident yeah ok excellent and the exponential comes because you have x dot divided by x yeah logarithmic so it is a linear system all linear systems of solutions which have exponentials ok nice and this is the x t zero initial condition so you see all our actors are in the play yeah you have the initial time you have the initial state you have time you have the state x basically is the solution so we have everything that we need here now once I if I fix a particular t zero so for a fixed t zero I will just denote this as some gamma yeah the exponential of minus six sin t zero plus six t zero cos t zero plus t zero square I denote that as a gamma ok because once I fix t zero this is a constant ok so that I can just look at the function of time then this is ok alright ok. First thing to note is that so what do we have to do? We obviously have to find a given an epsilon find a delta right this is our job yeah this is what we are trying to do so before getting to that I want to sort of understand this function if you look at this function this is true for this function that six t six sin t minus six t cosine t minus t square has to be negative for some large time ok why? t square quadratic dominates yeah everything else is linear yeah whatever you do it doesn't matter yeah the quadratic will dominate yeah the negative quadratic will start to dominate beyond a certain point yeah so the plot might look something like this that you have on the left yeah picture looks something like this so the good thing is it's a first of all it's a nice smooth function first of all yeah smooth continuous everything right this function inside the exponential is smooth yeah very good function right so it is smoothly changing so it's not like suddenly it can have a jump and it go to infinity and come back or do something funny like that no right so it's a nice function and I know that it comes it become negative after a certain point even if it started positive alright so I am guaranteed to have an upper bound m to this function ok and this upper bound will be between initial time and this finite capital T make sense yes yeah just look at the picture this picture is very very illustrative I mean again picture the real plot may be different there may be oscillations and all that given that there is sinusoids but more or less the idea is if the function is going to hit a negative value beyond a certain point so it's going to become negative beyond this point here it can only have limited exploration right by smoothness of functions and you know what is it central value theorem and all that yeah by all the nice results that you have for smooth functions if it's positive for only a finite time it can only a finite exploration on the y axis can't just go to infinity and come back ok excellent great so what is this I define m as exactly this the supremum from between this time of this function ok which is finite by continuity ok so continuity smoothness gives all the very very nice results yeah I don't have to you know bang my head so if you give me an epsilon what will I choose my delta S? I will choose my delta S epsilon over gamma e to the power m why? we will see very shortly yeah remember this I choose delta S epsilon over gamma e to the power m so I have repeated this here but remember keep this in your mind that gamma depends on t0 you ask me how the delta depends on t0 happens right ok I did not say how why my equilibrium is 0 in this case so if my x0 is less than equal to delta S which is equal to epsilon over gamma e to the power m correct by this expression of the solution I know that x t is less than equal to norm x0 gamma e to the power m yeah everybody gets this understand this right why? x0 is I am just breaking it into norm therefore I get a less than equal to right first I have the absolute value here and then when I take the separate pieces then it becomes a less than equal to anyway right just by breaking the absolute value I am just writing it as norm to be general I know that this is x0 I retain it as such this is gamma retain it as such this is exponential of this guy but the largest value this guy can take is e to the power m yeah so just plug in for delta here what happens? x t is less than equal to epsilon correct if you plug in for this delta in fact this guy goes here gamma e to the power m cancels so I just have x t is less than equal to epsilon yeah so I just proved it proved whatever I needed that x I wanted the state to remain inside the epsilon ball yeah but notice very carefully that my delta depends on the initial time very important delta depends on my initial time I cannot change that so all I have proven is stability in the sense of Lyapunov not uniform stability in the sense of Lyapunov ok because my delta depends on initial time ok also remember that it depends on epsilon also that is what the picture I kept making right you give me a small epsilon ball I will give you a small delta if you give me a large epsilon ball I will give you a large delta right so delta will sort of scale in fact linearly with epsilon in this particular case will scale linearly ok you can't ignore this yeah so not guaranteeing any boundedness ok great so everybody I hope convinced that this is stable what about uniform stability? Now I want this delta to make this delta independent of t0 so that is the question I ask here can gamma be independent of t0 because that is the only delta t0 dependence right nothing else depends on t0 only gamma so if I want to make delta independent of t0 I need to look for ways to make gamma independent of t0 ok. Now the important thing to understand is whenever I am trying to find a common delta that is independent of t0 I have to hunt for the smallest delta I hope this is sort of evident ok for example I mean I will just try to make a picture so suppose I give you this epsilon ball ok and then I give you some initial time corresponding to that I give you a t0 and corresponding to that I have a delta ball I would say this is delta ball ok. Now if I give you another initial time t0 prime then I get a delta prime ball right corresponding to different initial conditions I get different delta balls. Now the point is if I want a delta ball which works for both initial cut times for t0 and t0 prime which ball will I choose? The smaller one right obviously right because I want to stay inside epsilon so it has to work for both times t0 and t0 prime I have to choose the smaller ball if I choose the larger ball it will work for t0 but if I started t0 prime I am guaranteed to get out there is a potential that I get out not guaranteed but there is a potential that I escape ok.

So I have to choose the smaller ball so if I extend this to all possible initial times I have to find the smallest delta always if I want to get uniformity must find the smallest delta ok. So that is what is mentioned here ok. So I want to find the smallest possible delta which means I want to find the largest possible gamma in this case right ok. Now look at the expression for gamma I have just copied it for your benefit here what do you see? And it is written here but it is evident that it has the opposite behaviour of the t expression right. So this actually blows up goes to infinity in fact it will become positive after a certain time certain t0 and it will blow off to infinity because t0 square will dominate everything yeah.

So gamma t0 goes to infinity as t0 goes to infinity. So there is no upper bound on gamma at all there is no largest gamma yeah because if I choose gamma t0 as infinity then my delta is 0 which is to not allowed right delta has to be positive basically you are giving no initial condition right. So therefore no uniformity is possible in this case ok no uniformity is possible in this case because gamma is going to blow up alright and therefore it will not be uniformly stable you cannot achieve uniform stability in this case ok. So that is really the point so the system is stable but not uniformly stable ok. So this is how you analyse these properties we will talk about other properties in the upcoming class but this is sort of how you analyse your property you have to actually solve the system yeah.

You can imagine that this is not realistic yeah this is a very simple linear scalar system that I gave you and we did this bunch of pretty lengthy analysis to figure all this out. Once we solve the system you cannot expect that we are going to be able to solve most non-linear systems. Once I give you a vector system you will even a second order system you will struggle to get an analytical solution and any numerical answer is no answer don't try to ever give me numerical answers that I tried 100 initial conditions and I got that this delta works impossible not acceptable yeah because there may be that 101th example initial condition for which it will not work ok. So numerical answers not ok which is why subsequently not in the next lecture but subsequently we talk about the Lyapunov theorems which are actually better tests they are tests rather than just like your convergence test Lyapunov theorems are tests they are meant to give you easy conditions for doing this ok. So not actually use these definitions which is virtually impossible just like the supremum definition cannot use it yeah you can't just keep scanning to find supremum of a matrix.

So you have those simple formula here you have these Lyapunov theorems ok which will form the basis of everything we do in terms of design ok. Thank you.