Nonlinear Control Design

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Week 2 : Lecture 8 : Stability- Part 1

Alright, so we have you know sort of looked at lot of these preliminaries now, yeah. So we will very quickly jump to the real stuff, yeah. So, let us start talking about stability, okay. And I have written as you can see separately in the sense of Lyapunov, okay. So this was of course all of these notions are due to, the notions that we study are due to A.M.

Lyapunov, Prussian mathematician, probably non-linear control, the way we know it will not exist without him, okay. So maybe somewhere in 1860s, 70s and so on, he wrote couple of articles which you know which sort of delineated what is the notion of stability and how do you ensure that stability is achieved, okay. So very very I would say important contributions to the field, you know like I said the field itself may not have existed because this stability is basically the notion that we are always hunting as control guys, okay. Whatever you do, whatever system you are trying to dive, anybody who is working with any dynamical system, eventually most of your feedback and most of the control that you talk about, be it autonomous cars, be it you know aerial vehicles, be it smart grids, you are always trying to hunt for stability, yeah.

Because the basic idea is and you already know from linear system, you have the notion of input output stability, right and then you have internal stability but most people hardly study internal stability in linear system, we usually talk about input output stability, yeah but it is still a notion of stability. The idea being that external disturbances do not make your system deviate significantly from its operating point, okay. So once you achieve an operating point, for example you know your robot has converged to a trajectory that you wanted to follow, yeah, you don't want it to deviate you know if there are any disturbance, okay. So notion of stability is how you classify almost all problems in dynamical systems and control, okay. So there is no, of course there is notions of optimality which is a separate sort of line of thought in itself where you don't talk about you know being resistant to disturbance, being robust to disturbance and all that but then it is open loop, what we call open loop, okay in the sense that any optimal trajectory or anything optimal that you come up with and again trajectory is again a very general word, doesn't have to be something a robot is following or something a car is following, no.

Even a smart grid biological system you can always create trajectories, okay. For us trajectory is just a bunch of a smooth curve in state space, any smooth curve in state space is a trajectory for us, okay. So these trajectories that are designed with some considerations, maybe optimality considerations, yeah for example if you want a satellite, you are launching a satellite and you want it to go to the moon, once it is, the first thing you do it is you put it in a low earth orbit then you know it tends to expand its orbit, expand its orbit then after a certain point it has something like a swing, okay between the earth when the moon gets closest to earth there is a swing which is, which means that there is a thrust, extra thrust that is applied at a particular trajectory at a particular time. So and once it applies the trajectory it escapes the earth's trajectory and it sort of starts rotating about, it moves in straight line then it goes to the dark side of the moon then it starts rotating about the moon, okay. So there is like a particular escape route it takes from the earth's trajectories by doing a thrust maneuver at a very specific time, okay.

This is very optimal, this is like one of the purest applications of optimality you can see, okay. Here it is mostly its open loop, there is no feedback or anything. At that particular time when you see that you are, you have this earth moon sort of a very nice appropriately located earth and moon, you do the burn, you do the burn maneuver, you create the thrust, you escape the earth's trajectory, earth's orbit and then you move on to the moon's orbit, okay. So this is more or less standard way of how we fly satellites because otherwise we will never have enough fuel. If you did some random ridiculous thing you will never have enough fuel to reach anywhere, okay.

So I mean we have only limited fuel, it is not like we have petrol pump or something that is going to fill gas for us in between. So you just have enough, this is where all the optimality questions come in, okay. So this is an optimal loop, optimality is purely open loop. However, once you get into orbit, okay or you think of the lander problem, okay, you need feedback because there will be disturbances which are trying to, once you have an optimal trajectory, you try to follow the optimal trajectory, okay. You need feedback.

Why do you need feedback? Because you want to keep following the optimal trajectory. What if you deviate, what if you have, you know suppose you had some solar panel and then some serious solar radiation happened at that time, okay. Because some particular moment when you know some sun spot exploded, there was some extra solar radiation coming through, so you started, you know reorienting and tumbling in your orbit and doing some crazy things which you do not want to do, okay. You want to be pointed in a particular space. Then you need feedback, right.

I mean you have sensors which are picking up that, okay, I am starting to tumble, so now I do a detumbling control or any general attitude control, orientation control, so I sort of make sure that I come back to a particular orientation and I do this, okay. So that is where stability comes in, okay. This is feedback, this is stability, optimal, open loop, okay. Both are very important. These days there is of course concepts of doing them both together also and people are trying to derive stable laws by running optimal engines.

So that is also one way, but it is numerical way of doing things, maybe something that is more research than reality at this point. But yeah, our field relies completely on stability, optimality obviously there are quite a few courses in SysCon and otherwise also, you can, I would say you should always get exposure to both sides of the coin, okay. Alright, very very big preamble, I think very motivating. Alright, stability, okay. We are always talking about a system which looks like this, x dot is f t x, okay, with some initial condition.

Whenever I specify a dynamical system, I specify a initial condition, okay, without initial condition, yeah, does not make any sense. Then there is a solution, once I plug in the initial condition, usually denoted by a different symbol, okay. So in most mathematically precise text books like Vidya Sagar for non-linear systems, you will find the notation for the solution is different from notation of the state. Although in a lot of my notes you will find, okay, so fundamental matrix is more a linear system motion, yeah, so the state transition matrix comes from the fundamental matrix. So that is little bit more of a linear system motion.

In non-linear systems the terminology is different, the notation looks similar, okay. So this is called a solution. If I wrote this as phi t x 0, okay, just this, just change the notation to and put the t as the subscript, okay, this is called a flow. This is called the flow of this dynamical system as stressed, okay. Why is it called the flow? It is very beautiful, it is amazing, yeah, I mean how we have made everything very geometrically intuitive, yeah.

So just think about this. Here it looks like just a solution, right, I mean I plug in some initial condition, initial time, mostly whenever you talk about flows you sort of don't talk about the initial time, you sort of forget the initial time. Technically you should remember the initial time also but most often, more often than not you forget the initial time, you say that it is some fixed time t 0 and then you keep changing the x 0, okay. So here we are just talking about plugging the initial condition, getting a solution. So this is the solution, function of time but the flow is something way more interesting, right, it gives you something more interesting when I look at it in this form.

Why? I say I have a bunch of initial conditions. Say these initial conditions come from an ellipse which I call capital X0, okay. Now by virtue of this differential equation solution, once I plug in 1 X0, okay, and I flow it, flow it for time t, okay, just like you can think of flow in the river. You put one leaf at one point, another leaf at another point in the river, another leaf. So you put a bunch of leaves from this ellipse into the river and it flows along this solution, right, because once I plug in a X0 and I plug in a time t, I move in a certain way, right.

So what is this? I move here say, I move here, I move here. So it may so happen that I may get a little bit distorted, right, and basically what I am saying is this is time t, okay. So this is time t. So basically all these leaves, imaginary leaves that I put in this flow, they of course move differently, right. They cannot all be even though the average velocity of your stream may be similar and all that, but overall because of obstacles or whatever, everything flows differently and you may have a distorted shape now, yeah.

You may have start with an ellipse, you may have a distorted shape, okay. So this is the notion of a flow, okay, and a lot of controllability and observability notions are based on flows, okay. We do not, again, we are not sure if we will talk about those in the non-linear context in this class. I am not sure if we will have the time and it is also deeply more intense mathematically. So I do not know how much we will be able to prepare ourselves for it, but that is the notion of a flow, okay.

We basically just talk about the solution, okay. Why? Because we are at a lot of times interested in this function of time, okay, because we want to look at this as a function of time, alright. Once we put in the initial condition and initial time, we have a function of time here, okay, and we look at it. Sometimes we just call it x of t by the way, yeah. I do not actually specify this.

I write it as x of t, okay. So whenever I write it as x of t, please understand that we are talking about the solution, alright, great, great. Yeah, I know, it seems like we are talking too much about just some notation, but it is not because the solution is fixed only by the initial condition. Once I change this, everything changes, alright, great. Once I have a system like this, I need to talk about equilibrium. What is the equilibrium? The equilibrium is the state from which you never move, ideally, ideally, yeah, in reality you will always move, but ideally it is a solution from which you never move, yeah.

Very simple, if I have rolling object like this, I mean, in fact every point is an equilibrium for this, right. This is a very interesting example, right. Every point is an equilibrium, once I put it here, it is fixed, put it here, do not disturb it, fixed, right. So this sort of a system, everything is in equilibrium. This is an example of a non-isolated equilibrium, okay, because every point in x is actually an equilibrium, alright.

So equilibriums are, how do you compute the equilibrium? You compute it by equating the right hand side to zero, because that is what makes sure that x dot is zero. If x dot is zero, states are not going, moving anywhere, so you are fixed in state space, means, yeah, that is essentially what you want, you are at equilibrium. So equilibrium is computed by equating this to zero, okay. What is an isolated equilibrium? Equilibrium is isolated if there is no equilibrium arbitrary close to it, okay. I do not write it as a definition deliberately, yeah, because there is no need to make it mathematical.

All you want is, there cannot, if you have one equilibrium, no equilibrium should be arbitrarily close to it, okay. Then it is an isolated equilibrium. This is an example of a non-isolated equilibrium. This is an equilibrium, this is an arbitrary close to this. I have equilibrium everywhere, okay.

And this is also an example if you look at this right here. x one dot is x one x two, x two dot is x one square. What is the equilibrium? Equate these two, zero. All I need is x one to be zero, right.

All I need is x one to be zero. x two can be anything, right. I hope this is clear. Yes, I am equating x one x two to zero and x one square to zero. So once x one is zero, both are zero.

Nothing moves. So x two is arbitrary. So I mean equilibrium look like this. And what is that? If I draw it on the x y axis, it is the entire y axis, okay. The entire y axis is the equilibrium. This is a non-isolated equilibrium.

We don't like this, alright. We don't like this because all our results are based on convergence, okay. Now so stability, asymptotic stability, these are all properties which somehow connect to convergence. Now if you tell me that I am talking about the origin for convergence, you can't because when the trajectory comes very close here, yeah, so this is also arbitrary close to the origin, right. So basically what I am saying is there will always be a point which is so close to the origin that the talking about convergence of the origin and convergence of that point is identical, yeah. You will never be able to talk about convergence to the origin because you will always have points so close to it, equilibrium so close to it that talking about convergence of origin and talking about convergence of the other equilibrium is exactly the same, okay.

So you want to, in a lot of cases you can transform the system so that your equilibrium becomes isolated, alright. You may be able to do it, yeah. If not, you cannot talk about stability in the sense of Lyapunov in these cases, okay. I hope that is clear. You will not be able to talk about stability in the sense of Lyapunov if you do not have an isolated equilibrium, okay.

So please always verify that your system has an isolated equilibrium. If not, figure out a transformation if possible to convert the equilibrium to isolated equilibrium. If not, sorry, you can't do Lyapunov stability. You will have to figure out other motions of stability, okay, alright. And that brings us to the first notion of stability.

Let's see if we can highlight this, alright. So this is the notion of Lyapunov stability, just called Lyapunov stability, okay, or stability in the sense of Lyapunov, alright, okay, great. Remember we are going back to epsilon delta definitions, yeah. What does Lyapunov stability try to classify in terms of solutions? It says that if you start close to the equilibrium, you will remain close to the equilibrium. That is it.

This is what is Lyapunov stability, yeah. In words, it just says if you start close to the equilibrium, that is if your trajectories are initialized close to the equilibrium, that is x0 is close to the equilibrium, then x of t, that is the solution, will remain, always remain close to the equilibrium, always for all time, okay. That is Lyapunov stability. In, when you talk about, when you say Lyapunov stability, there is no notion of local or global. It is just Lyapunov stability.

There is no notion of local or global. All, you are not talking about convergence, notice. I did not say if I start close to the equilibrium, I will go to the equilibrium. No. I just said if I start x0 close to the equilibrium, my solutions xt will always remain close to the equilibrium.

That is it, okay. It is actually BIBO stability. Typically, it is sort of comparable to BIBO stability. It is bounded input, bounded output stability. Comparable. Not the same, okay. Comparable to bounded input, bounded output stability from the typical linear system sort of motions, okay.

Alright. How do we put it mathematically? We put it as a challenge solution, always like this. Given epsilon, find a delta. Remember, when we talked about convergence, we talked about given an epsilon, find an N, okay. So here, for all epsilon, this is the notation, for all epsilon positive, there exists a delta which can potentially depend on the initial time and epsilon itself positive such that whenever x0 is delta close to the equilibrium, xt is epsilon close to the equilibrium, okay.

So always start with epsilon, okay. Please never try to flip this. I always get this question. First, you are given an epsilon, then you find a delta, not the other way round, okay. Although the way I said it in words seem like the other way round, if I start close, I remain close but that is not how the mathematical challenges or mathematical definitions. Mathematical definition says, first you predefine how far you are allowed to go from the equilibrium.

Then I will give you how small my initial condition should be, okay. First you give me an epsilon, then I give you a delta such that if you start in a delta ball around the origin, you remain in an epsilon ball around the, sorry, equilibrium, clear. By the way, whenever I talk about this, I may very instead of saying norm and norm difference x0 minus xc and all that, I will keep saying delta close or delta ball, yeah. Please get used to this because we have already spoken about what is the, you know, norm, what does the 2 norm x, 2 norm of x less than equal to 1 look like, looks like a ball, okay. So whenever I say ball, does not have to be a ball, can be a square, can be a rhombus depending on the norm you choose, yeah.

So here depending on the norm I choose, notice I have not mentioned any norm here, yeah, these are all vector norms, this is also vector norm, but I did not specify 1 norm, 2 norm, you can choose any norm, okay, does not matter, norms are comparable, just do not change the norm. So important thing is you are given an epsilon, then you find a delta, okay, and I keep using the word delta ball, epsilon ball just to indicate norm x less than something, norm x less than 1, norm x less than delta, norm x less than epsilon, okay, please be aware. Can anybody tell me if epsilon ball will be larger or delta ball will be larger or epsilon greater than delta, epsilon less than delta, epsilon equal to delta, does this definition indicate any relation between epsilon and delta? Epsilon can be larger than delta, that's a very vague answer. Delta should be equal to epsilon, no, does not necessary, epsilon should be smaller, so you are saying that if I start in a larger initial condition ball, I will remain in a smaller final condition, forever I will remain at a smaller ball, okay. Let's look at all cases, what happens if, let's look at cases, right, I mean what happens if epsilon is less than delta, suppose you give me an epsilon and I give you a delta which is larger than epsilon, what happens? Can you check both conditions? This condition, this condition is obviously satisfied because I gave you the delta, right, so you will, this has to be satisfied, what about this condition? Actually, this is by the way, I am sorry, it's not evident unfortunately the way I made this, this is included in this definition, yeah, I will just do this, yeah, that's all, yeah, for all t greater than equal to t0 is already obviously included, okay.

So now if epsilon is less than delta what happens? For to this guy, what happens to this guy? Not satisfied at t0, if I put t0 here, the distance between these guys is delta which is larger than epsilon, so this is not satisfied at initial time itself, so there is a problem if epsilon is less than delta, okay, if epsilon is equal to delta no problem, yeah, yeah, but so this is not possible, no, this is not possible, okay, so epsilon has to be greater than equal to delta, it makes intuitive sense also, right, my initial condition ball will be smaller than where I want to remain for all time, yeah, I mean if I tell you that I want to remain in say, you know, I mean in a 5 centimeter ball for all, 5 centimeter radius for all time, my initial condition definitely has to be smaller than that, I mean much smaller for me to be able to because I have to allow for some expansion, I can't just assume that the system will, you know, remain inside, you know, even equal is difficult to achieve in most cases, okay, alright, great. So this is sort of the picture here that I typically show, so this will be the epsilon ball, the larger ball, corresponding to it you will always find a smaller delta ball, so that your trajectory start here, allow for it to get out obviously, I mean or remain inside but definitely can't go inside instantly, yeah, so delta has to be less than equal to epsilon, okay, so that is the picture. Thank you.