Nonlinear Control Design

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Week 2 : Lecture 7 : Signal Norms and Cauchy-Schwarz Inequality

Hello everyone, welcome to another session in nonlinear control. So we were looking at some preliminary material since last time. And so some of this, I mean, we didn't of course cover this material, we were looking at a few myths and temptations in nonlinear control. This was sort of one of the first things that we looked at basically, meaning to say that the function convergence doesn't mean the derivative convergence and the derivative convergence doesn't mean the function convergence and so on. This was sort of the first thing that we did. And then we moved on to some preliminary material on vector and matrix norms.

So obviously, we started with the vector norms. So if you have a fixed vector, then you can define these infinity norm or any p-norm in this way. And using these vector norms, we can in fact graduate over to matrix induced norms as well. So the matrix induced norm is defined using the supremum and the vector norms.

And of course, there are also simpler formulae in some sense for computing these. So we actually saw that there are these, you know, simple expressions for this induced norms. Of course, we have the Cauchy-Schwarz inequality on the induced norms. We have these very simplified expressions for the infinity one and two matrix induced norm. Otherwise, it would be rather difficult to compute the supremum and so on.

So we also looked at some properties of symmetric matrices. And in general, we looked at the notions of what is a normed linear space. So some more abstract content is what we discussed. So these were ideas on what is the notion of a vector space, a linear space having a norm, essentially a notion of length, if you may. And of course, we also showed proofs, we saw a little bit of the proof of, you know, when these particular norms that are defined actually satisfy the norm properties, right? Primarily the triangle inequality because the rest of the properties are relatively easy to verify.

And we saw those as well. And then we looked at what is the notion of, you know, convergence and Cauchy sequence and so on. And this led us to notions of, you know, complete normed linear space. So what is the meaning of it? So there is a vector space, and then there is the epithet that is a normed linear space or normed vector space. And then there is the idea of a complete normed linear space.

And that's what is called a Banach space. Essentially, in such spaces, the notions of convergence and Cauchy convergence become identical, right? Again, we saw examples, I mean, in Rn, which is the sort of vector space that we deal with for most of this course, of course, is a Banach space, right? So we also saw a more advanced notion of inner product space. So we looked at normed linear space, which is the idea of norm, which is the idea of length for vector spaces, for general vector spaces. And then one's also interested in operation between vectors, right? So how does one vector operate on the other? So that's one particular operation that's defined is an inner product, right? And so a vector space that is a normed space, or in fact, you don't necessarily have to norm, but anyway, the normed vector space, if it's endowed with the inner product is called an inner product space. And of course, the inner product also has a few properties, it has a symmetry property, distributivity property, scalar multiplication property, and the fact that the inner product of the vector with itself is non-negative and zero only if the vector itself is zero.

So then, just like the normed linear space, we looked at the completeness, we are also interested in completeness of the inner product space. And that's the idea of a Hilbert space. So what we say is that an inner product space, which is complete with the associated norm, right, it's evident that if I'm given an inner product, then if I operate, if in the inner product takes two vectors, as you can see, so if I put in the same vectors, x comma x, then I get a norm, right, it can be shown that this is in fact a norm. And so the idea is that if the vector space is in fact complete with this particular norm that's generated from the inner product, then what we have is called a Hilbert space. Again, as always, Rn is an obvious example.

So once we looked at this, we wanted to get signal norms, we sort of went over this in not so much detail. And so that is the first thing we want to do today is that we want to look at the signal norms in a little bit more detail. So what is the signal norm? See, until now, we've been looking at vector norms, and matrix norms, right? Signal norm is also a norm that's defined on a vector space, right? So, but the only difference is, we are not talking about a vector signal, which means that it's a map from time to Rn. And that's what most of the states that we will be looking at subsequently are, right? Eventually, once you solve a nonlinear differential equation, what you get is a function of time. And it's typically a vector function of time, because you will have multiple states, and you will have more than one state.

So you typically have a vector function of time. And so what we then do is we define these signal norms, right? So the P signal norm, which is denoted in this way, is defined using the vector norm. Please note that the vector norm contains the time argument, right? Because it has to, because the vector is the vector norms are for fixed vectors. So until I fix a time t, x of t is not a fixed value, right? So therefore, in order to evaluate a vector norm, I will need to specify the time. Therefore, whenever I'm talking about a vector norm, for a signal of a time varying quantity, I will have the time argument in here, right? It's almost like saying that I have a signal, and I'm looking at the value at some particular time, right? So that's the idea.

So using this vector norm, and note that this is arbitrary vector norm, we didn't say that it's the one norm, two norm, infinity norm or anything, because you see no subscript here, right? So this is flexible. And so we take this vector norm, take its, take its power to the power p, and integrate from zero to infinity over time, and then take one over p, right? That's what is the p signal norm. Yeah, p signal norm. Similarly, we have the infinity norm, which is defined slightly differently using the supremum, it just says supremum over all time greater than equal to zero norm, vector norm of x t. Okay, now, as I said, the vector norm used is arbitrary, but typically, for a single problem, for evaluating a single particular problem or a control question, you would always use the same vector norm for all the vectors you have, okay? Otherwise, you will end up getting really ridiculous results.

Yeah, it's important that you're consistent, you use the same vector norm everywhere. But the choice of that particular vector norm that you use for the entire problem is completely free. You are free to choose which vector norm you want to use. Okay, great. Now, so, so obviously, the that's what we are saying here, the choice of the vector norm is not matter, doesn't matter, but do not switch, be consistent throughout the problem.

Now, one of the important things that we define here is that if a particular signal norm is finite for any given p from one to infinity, then we say that x belongs to this capital script L p space. Say that x belongs to this script L p space. Okay, this is a very large class of functions. We just called the L p class of functions. Okay, and these are very important classes, they appear everywhere in analysis, Fourier series.

These are essentially some kind of advanced integrability type conditions, because, as you can see, each of these norms is defined using integral of some power of the vector norm, right. So these are like integrability conditions. So if you take p equal to one, this looks like classic integrability condition, but if you take p equal to two, and so on, they're just advanced versions of the same integrability condition. Okay, so please remember that these are these these define a very, very large class of functions, and very, very large and very, very useful class of functions.

Okay. So one of the important things that we realize immediately is that when we say that x belongs to L infinity, we are just referring to bounded signal. Why? Because the infinity norm is just defined by the supremum over all time. Yeah. So it's easy to see, easy to prove something like this. How do we go about it? If x is bounded, if x of t is bounded for all time, it means that there exists some constant m, such that the vector norm of x for a particular time t is always less than equal to m for all t.

And this is true, this holds for all t, right. If you fix up time, then the vector norm x t is always going to be less than equal to m. Okay. Again, this m may vary depending on which vector norm you chose, but there exists such an m. So we don't have to worry because we are going to be consistent.

We are going to use the same vector norm all the time. All right. Great. Now, once you know that the vector norm x of t is less than equal to m for all t, the supremum also has to be less than equal to m. Because if at every instant in time, I evaluate the vector norm and it's less than equal to m, then the supremum also has to be less than equal to m.

Right. The supremum is nothing but the least upper bound. So I'm saying m is an upper bound for all time. Therefore, m also has to upper bound to supremum. Right. Which means that there is a bound on the supremum norm or the infinity norm.

Therefore, x belongs to L infinity. Right. So, you know, looking at the other side of the argument, if the function, you know, if I say that the infinity norm is in fact equal to m, right, now, then I know that supremum is equal to m, which means for all time, the vector norm x of t is going to be less than equal to m because again, infinity norm or supremum essentially is the least upper bound. So it is in fact an upper bound, whether the least or the largest, it doesn't matter, but it's an upper bound for the signal. Therefore, this upper bound will always hold.

And this indicates that the signal is a bounded signal. All right. So it's a very easy proof. And you can claim that signal is a bounded signal. So like I said, the LP space appears in quite a few places in mathematics.

Anyway, so LPs, typically the LP can be seen as a regularity condition. Yeah. And typically appear in the several convergence type results that you will see. And small LP is a discrete counterpart. So if you have not a continuous function of time, like we are using, right, but a discrete function of time, that you just have the function value at step one, step two, step three, step four, and all that, then and so you use summations instead of integration, then you have the small LPs.

And the same notions apply there as well. Right. Now, as far as the notation goes, let's be careful. The vector norm, like I said, it's frozen in time signal, right, because the vector norm can only be evaluated if your function is fixed.

Right. So therefore, the vector norm, right, will always contain the time argument in there, right? It's a time frozen quantity. On the other hand, the signal norm, if you notice, either I take a supremum over all time, or I take an integral over all time, which means that the time argument goes away, vanishes from this quantity. Therefore, in the left hand side, there can be no time argument, it would be ridiculous to say that X t norm of p, right? So therefore, the signal norms will always have no time argument, just something like this and a subscript, maybe. Okay. Now, one of the things that we sort of know about vector norms is this notion of a norm equivalence.

For vector norms, we have this very, very nice result, which essentially says that if you take any two vector norms, they're comparable by a constant. Okay, what does it mean? If I take the q norm, then I can always find constants alpha beta such that I can compare it with a p norm like this, I can bound it on both sides with the p norm with using constants alpha and beta. Now you can see I can always flip this argument, right? I can always say that X is greater than equal to one by beta, Xp is greater than one by beta Xq. And similarly, X is less than one by alpha Xq. Therefore, the p norm can also be bounded on both sides with the q norm, right? So this non equivalence is very standard and holds for vector norms.

However, such an equivalence is not possible for signal norms. Okay, this in short sort of means that if I take any signal, that is any any vector function of time, then there is no guarantee that if it is belonging to an l1 space, that it will belong to l2 space, or if it is in infinity space, it will belong to l1 and l2 space. There is no such guarantee. Right? So these are completely distinct class of functions in general is what it means.

Yeah. And where does the problem come? Because you're sort of trying to integrate over all time, or you're not trying to take supremum over all time. This is this is where the problem arises. And let's see some examples of this. So the first very, very standard example is this function vector function xt is cosine t sine t. Right? And what is the infinity norm? So I'm going to take, it's my choice what vector norm I choose, I choose to take the two norm because you can see that the two norm is very easy to compute in this case.

So, so the infinity norm is the supremum of the two norm over all time. And what's the two norm? It's just one. Okay, so the supremum is actually equal to one. And the supremum is actually equal to one, as you can see here. Now, this means that the supremum is bounded, therefore the infinity norm exists.

Yeah, therefore x belongs to L infinity, as per our definition, right? If a function has a has a finite LP norm, then it belongs to the LP space, right? It has a finite infinity norm, therefore it belongs to L infinity space. Yes. Now let's evaluate the one norm or the x one. How will it look like? In this case, instead of taking the supremum, you're just integrating from zero to infinity. And this quantity is still one, because nothing has changed.

I've still taken the two vector norm, right? I've chosen to take the two vector norm, I'm choosing to be consistent. Therefore, the two vector norm still evaluates to one. However, now if I integrate this one from zero to infinity, then I get infinity. Therefore, the one norm is not finite anymore.

Therefore, x does not belong to L one. So therefore, there's no way you can propose any kind of norm equivalence like this, because one quantity is finite, another quantity is infinite. Therefore, there's no way there can be an equivalence, because there can be no such constants relating a finite quantity and an infinite quantity. There exists no such constant, right? Just pretty obvious that signal norms are slightly more evolved or involved notions, where this sort of norm equivalence kind of things don't hold. Okay. So the only thing the norm equivalence basically says for these kind of examples is that, you know, instead of the two norm, yeah, if I chosen some other norm, say some five norm or three norm, nothing would have changed, the constant would have changed a little bit.

That's it, right? There's a constant here would have changed. That's it. And that's what we are saying with norm equivalence here. Yeah, it did not it doesn't mean that this will not be infinity, so it still have been infinity.

Okay. All right. So let's look at some other examples. I mean, we just looked at an example where the function is bounded or L infinity, but not L one. What about the other cases? What about a function which is L two and not L one? So this is one such example, where a function is f of x is defined as one over x for all x greater than equal to one and zero otherwise. So let's evaluate. I mean, this is a scalar function. So there is no choice of vector norm or anything like that.

The norm is just the absolute value. So what is the L two norm? So if I want to say that I want to compute the two norm, then I will just have zero to infinity f of x absolute value squared to the power half. Right. Okay. And what is f of x squared? So this will actually reduce to what? This will just be one to infinity one by x squared dx to the power half.

And you already know this is nice, it's minus one by x. It's just minus one by x evaluated from at one and infinity and to the power half. And this is basically one. Yeah, so the two norm is just one. What about the one norm? What about the one norm? This is a problem. So the only difference that will happen is that this will become one to infinity one by x dx.

All right. And this is a problem. Why? Because this is going to be log x from one to infinity. And this is infinite. So therefore, this is not right.

So f is not in L one. Okay, I hope that's evident. Okay. Similarly, you have another example, which is where you have this. Just a second.

Just remove this. So we have a clean place to write. Alright. So yeah, so this is a case where you have a function f, which is in L one but not in L two. So actually, this should be the other way.

Just a second. I will fix this. L one not in L two. Okay. So again, not difficult to evaluate, I guess that this function is L one not in L two. Again, you integrate for the one norm, you're going to just do zero to infinity. And you have f going from and this will actually reduce to this is zero to one. Right. Interesting thing is this is only from this is not actually at zero, but at one, okay, but we still do this integral like this.

So this is just one over square root of x dx. Okay. And this will become I believe this will become two square root of x zero to one this is going to become two. Yeah, I think that should be fine. Yeah, I think that should be fine. Anyway, we can check is just this factor of two that you have to check.

But otherwise, I think this is fine. Yeah. And if I do the f two, now what happens is zero to one, one over x dx to the power half and again, this will land you in trouble, because this will become log of x zero to one. Yeah, and the problem is at x equal to zero, this is undefined, right? It's minus infinity.

Right. So this is again going to go to infinity. All right. So that's a problem again. All right. So that's not. So these are just some nice examples, right? That works for one space, right? It's an L one or not an L two, A two not an L one, L infinity not in any LP, any other LP and so on.

Yeah. So you can you can create many such counter examples, right? So basically to indicate that norm equivalence does not hold in general, right? So in signal, in the case of signal norms, and expected right not since we are talking about general, much more general norms. So what I wanted to look at is, since we have looked at so much of the norms, we've looked at nonlinear spaces, we've looked at the idea of the fact that the norms follow triangle inequality, obviously, one of the key properties of norms is triangle inequality. But one of the other properties, and then you also saw it for the matrix norms, is this sort of Cauchy Schwarz type of an inequality that the norms really follow. Yeah. So, so this Cauchy Schwarz inequality, of course, we stated it without any proof here for the matrix case.

But I wanted to work out a simple proof of Cauchy Schwarz inequality for the general case. So, so this is the notation that you have two vectors x and y, which belong to a normed linear space. So there's a vector space x and there's an inner product, sorry, in fact, it's not a normed linear space, it's an inner product space. Right? And the inner product, and this is in fact, the Cauchy Schwarz inequality that we want to prove, right? For the matrix case, you've already seen that, and this is the more general one.

So this is a very nice nifty simple proof. So if you take any vector u, it can be written as these two components, that is a component in a direction of some vector v, and something orthogonal to that. So how do you get the component along v? You just take the inner product and divide it by norm v squared. So essentially, it's the inner product is seen as a projection. Right? The inner product is seen as a projection. And then you have some vector w, which is orthogonal, right? Which is orthogonal to v, right? So you can always break any vector in these two components, if you may, in these two components, a component in the direction of any arbitrary vector v, and something that's orthogonal to it, right? We are not even saying that this is, you know, we're not trying to even define what this is, you know, more explicitly, because we don't need it.

Right? Now, if I take the inner product of u with itself, then you can, you can just write this formula again, right? So basically, this is u. So I just repeat the same thing here. And then I expand it using the inner product ideas, how inner product works, right? So from the first two, I will have u v over norm v squared whole squared, right? Basically, this multiplied by this, and then I'll have a v inner product with v. And then I will have a mixed term, which is twice u v divided by norm v squared, I believe they should be norm v squared. Right? Because of that, it should be norm v squared, and then there will be a v inner product with w.

And then you have a w inner product with w as the last term. Right? So now, anyway, we are not too concerned about, you know, what this is going to look like, and so on. This last term, but we know that this guy is going to zero because they're orthogonal. Remember that I could have expanded this term very easily as well. If I if I simply say, I mean, if I instead of w, I use, you know, w bar, and and then I say that I take right, so this would be this term. Instead of w, I could have simply written this term, but the point is that u and w are still orthogonal.

Okay, so that would have still been the case. Sorry, sorry, not u and w, but v and w are still orthogonal. And that that would have still boil down to the same expression. So that's why we're not expanding, but we could have if you wanted. Right? All right, great. So so basically using the fact that any vector can be written in components of a vector v and something orthogonal to it.

Okay. And this orthogonality is being defined by the inner product. Okay, is defined by the inner product. That's it. I mean, it's not necessarily 90 degrees or anything like that. It's being defined by the inner product.

Yeah, you may choose some funny inner product for which it is not 90 degrees. Now, once we have that, we know that this is a non-negative quantity. It's a norm omega squared, a norm w squared. Similarly, this is norm v squared. Right? And this norm v squared will cancel with this norm v to the power four, two, so I am left with norm v squared.

This quantity is greater than equal to zero. So what do I know? I know that u square is in fact greater than equal to just the first term. And I immediately get the Cauchy-Schwann's inequality. Yeah, exactly as I wanted. In fact, in Rn you can do something even simpler.

You have this triangle inequality, right, from the typical norm in Rn. And then you just expand both sides. Right? Just you take a square basically. You take a square and then you expand this side, you get this. On the left hand side, you have a norm u plus v squared, which is basically this quantity in an inner product space.

Yeah, you are using the norm that is being induced by the inner product. Right? And so you get what? u u, inner product of u u, inner product of v v, plus twice u v. Right? And so this guy cancels with this guy, this guy cancels with this guy. And so I'm left with and the two cancels out here. And so I'm left with u v less than equal to norm u times norm v. Alright? So this is how you would approve the Cauchy-Schwann's inequality in general. Thank you.