Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 12 : Lecture 69 : Finite time stability: Part 2

Hello everyone. Welcome to the second lecture in this week on finite time stability. So what were we looking at last time? In the previous bit, what we've seen is the need for looking at non-Lipschitz systems or systems which have non-unique solutions at the origin. So that's how that's what we formalized that the Lipschitz property essentially means that solutions exist and are unique. However, because we are looking at notions of finite time stability, it is not quite possible to have unique solutions from the origin, especially in backward time. And therefore, we do not require it even in forward time.

Alright, so that is why we settled on the notion of having uniqueness in forward time everywhere, but at the origin. So this is the notion that we choose to work with while defining the notion of finite time stability. So that's the system we were working with. It's an autonomous system, there is no explicit time dependence.

So it's of the form y dot equals f of y. And on top of that, we have that f is continuous in the entire domain containing the origin, f is locally Lipschitz in the domain, but excluding the origin itself. And f0 is 0, this is simply the standard assumption that is required to ensure that 0 is the equilibrium of the system. Now, this essentially guarantees that solutions exist for all initial conditions in the domain. And this Lipschitz condition essentially guarantees that the solutions are unique for all initial conditions in the domain, but at the origin.

So we don't want to deal with the origin, because there is possibility of non unique solutions at the origin if you are demanding finite time stability. Now, we also defined the notion of finite time stability. And how do we define it? We say that the origin is finite time stable, if there exists a neighborhood n, which contains the origin inside the domain D and a settling time function, which maps the neighborhood to some time 0 to infinity. And the settling time is exactly what you would understand from it, it is the time at which the solutions go to 0. So obviously, as you can imagine, if you start at the so this settling time function essentially maps initial conditions to the time in which these initial conditions go to 0.

So obviously, if you start at 0, since you are already at the equilibrium, you would expect the settling time to be nothing but 0 itself. And we also want that Tx0 actually converges to 0, as the initial conditions converge to 0. So notice that we never say equal to here, because we do not want to look at initial conditions at 0, because our assumptions are on sets that exclude 0 or the origin. Yeah. So therefore, we always talk of convergence to the origin, and so on.

Okay. So the second requirement is, the second requirement of finite time stability is exactly the convergence, the finite time convergence, it says that if your initial conditions are in fact in this neighborhood, excluding the origin, then we have unique solutions, right, which depend on initial conditions. On this time interval, it is closed on the left and open on the right. So it is from 0 to Tx0. And it is important to note that the solutions are never 0 in this interval. But as time goes to this value, yeah, we don't say equal to again, because this is not part of the interval.

So as time goes to this value Tx0, we say that x actually goes to 0. Okay. All right. So again, we are carefully excluding 0 from the mix. Yeah, we still want to converge to 0.

But we never want to start there. And so on. I mean, if you're starting there, that's considered as a special case where you're starting in the equilibrium and staying in the equilibrium. So there is no notion of I mean, we don't have to, the finite time that we get is 0 anyway, right, the finite convergence time that we get 0 anyway, that's what is encapsulated here. But we do not want to start at 0 otherwise.

Okay. And we are always looking at converging to 0. So limit also means converging to 0. And the last requirement is obviously Lyapunov stability that I'm not going to repeat, we've already seen that Lyapunov stability is a requirement of asymptotic stability as well. And therefore it is but natural that finite time stability notion would also have Lyapunov stability as one of the requirements.

Okay, great. Now that we understand finite time stability, we have a few conclusions that can be directly obtained. Again, I'm not going to prove any of these things in this course, this is a very short introduction to finite time and sliding, finite time stability and sliding mode control. So we are simply going to state a few results. So the first sort of proposition, which is an almost like saying an outcome of finite time stability is that is what we state here. Zero, if zero is finite time, I will just say if zero is finite time stable for one for the system that we already considered.

And let and we therefore have this, you already have this neighborhood n and settling time function t corresponding to this finite time stability definition, then what is understood is that a the solution is what is understood is that the solution is uniquely defined. And most importantly, it is understood that x t zero x zero is exactly equal to zero for all t greater than equal to t x. Okay.

All right. Yeah. So once you essentially how, how one would say this is that is uniquely defined for all x zero in the neighborhood n. Okay, what this is sort of trying to say is that once you fix an initial condition, you have a very uniquely defined path. And the important thing to remember is that, I mean, it's sort of understood just from the second piece here that once you that you will reach the origin in this time t x zero. And what happens beyond time t x zero? Well, you are already at the equilibrium. So you're not going to move unless there is some disturbance.

But we are not considering the disturbance cases here. This is for the purpose of defining for a disturbance free or a noise free situation, the notion of finite time stability. Therefore, once you reach the origin, which is an equilibrium, we're never going to move. Therefore, that's what is essentially said here, that once you reach the origin, you will always stay there for all time beyond.

It makes sense. So this is not a very complicated motion. Okay. The next thing to remember is that the solution the way this is written is has to be carefully written is has to be careful here. Actually, there's a dot here, which means that the purpose of writing this dot is it's uniquely defined with respect to the time. And in here, we say that x t the solution is actually this is fine.

I'll just keep the time here. It's evident that it's uniformly defined in time, right. And here I'm going to say it's continuous for all x zero in a neighborhood of the origin.

uniformly in time. Okay. So the first one says that the solutions go to zero and will remain there forever. The second one says that the solutions are continuous in the initial conditions. Okay, this is important. Okay. And the final assertion of this proposition, proposition is that t the function t of x zero is unique and again continuous in x zero.

Okay, so these are the important sort of outcomes from the definition of finite time stability. Okay. Now, what do we want to do? Obviously, we've always seen that these notions of asymptotic stability, these definitions never really helped us, right. So what we really want to do is to sort of have some Lyapunov like conditions. Okay, so that's really what we would like to do have some Lyapunov like conditions to characterize finite time stability as well.

Okay, and so that is sort of what we are moving towards. Right. So let's see how to state this. I'm wondering if all right, that's fine. So I will just call this Lyapunov like characterizations.

Okay, I'm just looking at Lyapunov characterizations for finite time stability. Okay.

All right. Let's see. Let's see. Now suppose I'm going to start with a more, well, I'm just going to directly jump into the simpler case, where and we already know the notion of V and so on and so forth, the Lyapunov function itself. So we define as before V dot of x as del V del x f of x for a V in C one, right. So if you have a continuously differentiable function V, then you can actually make this kind of a definition for V dot.

Okay. All right. And this, by the way, this works even for this sort of special case where you have solutions which are unique, but not at the origin and unique in forward time. All right. So this Lyapunov derivative idea is still valid here. Yeah, that V dot turns out to be exactly this.

Okay. All right. So, so what we need to understand is that important V dot is well defined in D removing zero, right? Because our solutions are well defined in D removing zero. Okay. All right. So what is the main result? So what is the main result? This is, let me see, Lyapunov, I'll just call it Lyapunov theorem, right? I'll just call it a Lyapunov finite time.

Theorem. And what does it say? Suppose there exists a V which is continuously differentiable and has the following properties. V is positive definite. Right? This is a standard requirement. I mean, V is just seen as a function of some X. Therefore, we just need to verify the positive definitities of V.

We already know how to do this. Next, V dot is continuous. Well, V dot being continuous is already evident from the fact that V is C1. So V dot is negative definite on the deleted neighborhood, right? On the deleted neighborhood of the origin, right? So it's negative definite, but you need to verify only on the deleted neighborhood of the origin. So, yeah, yeah, so you don't need to. So if you remember, negative definiteness had two properties that you check, has had two conditions to verify.

One is that it is the function is zero at zero, and it is strictly positive for all non-zero values. Okay. So in this case, you just need to check that it is strictly negative for all non-zero values of the state.

Okay. And that's it. Okay. And finally, we want to, you need this special condition, right? That there exists K positive and an alpha in 0, 1 and a neighborhood V inside D, right? Of origin. Such that V dot plus K V to the power alpha is less than equal to zero on this V deleting the origin. Okay. If you have this, if you have these properties, then apologies.

Then zero is finite time stable. Also, you can actually find or upper bound the settling time function, right? As T X zero is less than equal to one over K one minus alpha V of X to the power one minus alpha for all X in N, then N is as defined in the finite time stability definition. Okay. Now, it is important to sort of try to understand this result, right? I mean, the first one, I mean, the Lyapunov finite time stability theorem is not too different from the typical asymptotic stability theorem. In fact, the first and second look rather similar, right? So we require that V is positive definite and V dot is negative definite on this deleted neighborhood. On top of that, we have this kind of a funny convergence type of a condition, which is that there exists some positive scalar K and some exponent alpha between zero and one strictly inside zero and one.

Yeah, never one and never zero. And a neighborhood V inside D of origin such that V dot plus K V alpha less than equal to zero. And if you have these three conditions, we are in fact claiming that you have finite time stability. Okay. In fact, if you would understand that from A and B, you would be immediately able to claim some kind of Lyapunov stability. Yeah, I'm not claiming, I'm not saying that I'm giving a proof of this theorem, but I'm just trying to indicate why this might work.

Yeah. So from A and B itself, it's not difficult to see that you would have Lyapunov stability. The only thing that's left for us is to conclude the finite time convergence. And for that we can actually focus on this third statement. If I try to look at this sort of third statement a bit more carefully, yeah, it basically says that I have V dot is less than equal to minus KV to the power alpha. And if you see this is a scalar differential inequality.

Yeah, it's not a differential equation, but it's a differential inequality. Right. But it's not too difficult. I mean, you can deal with this in an exactly similar way, as you would deal with an equality.

Right. So this will be something like, and again, this is because of the fact that V is positive definite, that we can all do this. So V is scalar value, so it's actually a scalar differential inequality. So I can actually do things like this and integrate both sides. And if you see V alpha is less than one, right, so I will get something like, I will get something like V one plus alpha, I guess divided by one plus alpha.

I am just trying to see if I am doing this right. No, I apologize, this will be V not one plus alpha, but one minus alpha because alpha is in the denominator, right? This is one minus alpha. And this is going to be less than equal to minus K T. Yeah. And so if I solve from here, for the value of V, what am I going to get? I'm going to get V is actually less than equal to minus K T or minus K times one minus alpha T.

And this to the power one by one minus alpha. Right, this to the power one minus. Right, right.

I believe that's okay. I believe that's okay. Right. And because, so this is one over one minus alpha, this is fine. And you know that one minus alpha is strictly positive by assumption because alpha lies exactly between zero and one. Right. So this is strictly positive.

And therefore, this is strictly positive as well. In fact, one can very easily show that one minus alpha lies between zero and one as well. Right. One minus alpha strictly lies between zero and one.

Okay. So this is something that is right. Right. Okay. And now, if I want to, if I try to equate this to zero, right, suppose I equate this quantity to zero, what happens? Okay. All right. So, okay, so I see that the expression here has been given in terms of the V itself.

All right. All right. Let me actually try to see. So the left hand side would have more terms. That's what I'm missing here. So what I'm missing here is like a V zero.

Right. So where V zero is equal to V at x zero. Right. So in fact, I'm going to erase this and rewrite this carefully because I'm going to have V to the power one minus alpha is less than equal to k V zero. Yeah, that's what it makes sense. Right.

This has to be V zero minus k one minus alpha to the power t. Correct. Correct. Correct. Correct. This is exactly it. And now if I make sure that if if this equal to zero happens, then I know that V is obviously positive definition can never be less than zero.

Right. So I know that we will exactly be zero. We will exactly be zero. Right. And if V is exactly zero by positive definiteness, I also understand that my states will also exactly be zero. Right.

So that's sort of what I'm trying to. Yeah, yeah, exactly, exactly, exactly. So this is exactly what it is. One minus alpha. This was also one minus alpha.

So I should not miss this exponent here as well. Right. And now from here, from this condition, I know that the t I need will be actually equal to V zero to the power one minus alpha divided by k times one minus alpha. Right. And that is exactly this bound.

Yeah. In fact, this should be not X, but X zero. Yeah, there's not X, but X zero.

Right. It is a function of X zero. So that's exactly what we get from here. Okay. So this so obviously this is the time we get and this is obviously conservative because all Lyapunov analysis is conservative. We already understand that. And that is why we say that the settling time is always upper bounded. You never arrive at an exact value of settling time function, but you usually always almost always arrive at a upper bound for the settling time function.

Okay. So I hope you understand now that from the first two, you get the Lyapunov stability. That was the first requirement. And the next requirement was, of course, that your finite time convergence, which you get from here. And in fact, the expression for the settling time function is also rather easy to obtain just by integrating this scalar differential inequality condition that we have written here. And just by integrating it carefully, I was of course not doing it properly to begin with, but now that you've done it carefully, there is also the value of time t and the value at initial time as zero and then divided by one minus alpha.

So this is this integral evaluates to this in fact, right? So this will actually be from, yeah, if you are being careful, this will be from V zero to V and this will be from zero to t, right? This definite integral. And that's what we've done here now. And you just get a minus kt. And once you have that, you know that you just need to equate this guy to zero, because that means that this is going to be exactly zero because V can never be less than zero.

And once you have that, you know that the t can be calculated like this. All right. Okay, great. So we are left with a little bit of the converse theorem and we'll look at a simple example in the next session. Thank you.