Nonlinear Control Design

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Week 11 : Lecture 61 : Barbalat's Lemma

So, welcome to Non-linear control. We have been looking at feedback linearization for the past well maybe couple of weeks I guess. May have seemed longer to you, but it has been maybe couple of weeks. So, we did a little bit of the proofs. I of course did not prove the key result. I left it for you to read or maybe you know I do it in the end if there is time and so on or we can some of us can discuss it and so on.

But we looked at the key applications and that is of the Frobenius theorem which is essentially saying that involutivity and you know complete integrability of this distribution is equivalent. So, this is what we have sort of seen in the Frobenius theorem and we were looking at how to use it. So, for the fully feedback linearizable case it sort of gives us a very nice set of partial differential equations. So, that you can actually identify this control.

So, we actually looked at it specifically for the DC motor case. So, there was this DC motor dynamics slightly different from this of course there was this additional thing here which is essentially this guy and there was this. So, basically it is an fx plus gxu we have been looking at single input systems just to make things easier for us and essentially we were required to check only two conditions. First is that this g add so add f0 g add f1 g add f2 g all the way to add f n minus 1 g is supposed to be linearly independent. So, in this case n minus 1 g is just add f2 g.

So, this was one thing that was that we were required to verify. I believe we were able to verify this one I guess. So, we did g fg and then add f2 g was the one which was sort of complicated I am not sure if we actually verified this right we said that we will do some of this offline or even numerically for that matter yeah and then the second one was that we wanted to check the involutivity of add f0 g add f1 g all the way to add f n minus 2 g distribution made by these vector fields right and so we did in this case add f n minus 2 g is basically is add f g and this was basically very easy to you know verify. So, we had g and add f g and we just want to check if this is involutive right. So, all we had to do was just see if g and fg lie bracket is it turning out to be in the distribution itself or not and so we actually verified this that g and fg when you take the lie bracket of the two it turns out to be 0 I believe right that is what we got.

So, let us see let us see yeah yeah that was fg was this g was this I actually very so yeah this was add f2 g which turned out to be very complicated so and then we were trying to verify that g and f of g fg lie bracket is in the distribution this is very easy because this turns out to be I believe turns out to be 0 no yes this turns out to be 0 correct this turns out to be 0 and therefore this is trivially true that that this is in fact in the distribution because if you take any vector space 0 is in the vector space obviously right. So, therefore we were able to verify the involutivity condition once we had these two conditions we know that it is fully feedback linearizable and then all we have to do is to to sort of use this equality which is basically saying that you know the the you know the d beta whichever whatever is their output because now we are trying to figure out the correct output with respect to which with the system is feedback linearizable right and so let beta be that output so all we are saying is that this d beta is linearly independent so therefore this d beta product with vectors of the distribution is exactly 0 right and this is just an evaluation sorry yeah it is actually evaluating these you know partial derivatives here so you get a bunch of equations in partial derivatives and at this point you can pretty much you know you you get a few conditions you get that partial of B with respect to x1 is 0 so therefore there is no dependence on x1 so we have only dependence on x2 x3 and then once you have dependence on x2 x3 you can use the second equality to conclude that beta comes out to be something like this right. So, if you remember we had looked at the DC motor example earlier also and we had sort of guessed some outputs yeah and how did we do that we sort of we were only looking at partial feedback linearization yeah because we started with say person output x2 and then we only got a degree 2 not relative degree 3 therefore the system was not fully feedback linearizable with x2 and so on and so forth so we tried different things and then we just tried to find a so basically what we were doing was we had fixed sorry not x2 but x3 we had fixed the output okay here with this knowledge of Frobenius theorem and the lie brackets we are going backwards we are trying to identify what should be the output y with respect to which the system can be feedback linearized fully feedback linearizable and so in this case it turns out that this is in fact that output okay even if this is unintuitive and whatever I mean it may not be something that's making any physical sense to us but this is what it is okay all right. There is this small little space rigid body example that I have also sort of done here I am not going to cover this I want you to take a look at this on your own yeah I have asked for this output with respect to which you get full state feedback linearization and in this case it turns out that I mean I have actually solved it you can take y equal to rho itself which is the kinematics parameters right it can be the MR modified Rodriguez it can be quaternion whatever I believe this is written with respect to the modified Rodriguez parameters yeah this is the rigid body equations you have in the current homework also I believe right so then basically with this output it turns out that you can get you know your adequate E-arised or the feedback linearized system alright under certain conditions of course right say anyway so I have I have actually done some computations and so on and so forth I will leave you to look at this on your own okay so that sort of brings us to the end of what we want to do with feedback linearization as the TAs have announced you have a tutorial right you will do a little bit more I have in I mean I have instructed them to sort of bring some interesting problems where you actually computing these lie brackets because that's the challenging part just guessing an output and keep taking derivatives is easy because that's not difficult right you take an output and then you keep taking derivatives wherever the control shows up you get the relative degree then you guess the rest of the states that's still okay yeah but this is a little bit more complicated but this has a little bit more general applicability therefore I've asked them to take up some interesting examples you are also free to bring your own examples yeah and try to discuss it in the tutorial yeah that's fine all right okay great now what we want to do is I'm actually pulling out a little bit of what I taught in adaptive control because I'm going to give you now we are essentially more or less at the end of the standard design methods yeah there are no more generic standard design methods okay everything else is very specific to systems and so on so you have learned until now Lyapunov redesign right basically take a Lyapunov function or a control Lyapunov function and try to identify a control by taking derivative of the control Lyapunov function along the system trajectories that was the first one then we went to back stepping which is basically how to construct these control Lyapunov functions sequentially right and then we went to passivity based ideas where if you have some passivity inbuilt in the system then you have a certain structure you can actually come up with the nice storage function you can take the storage function and come up with you know nice Lyapunov functions right so you also have this passive interconnections and things like that so we did the passivity based methods then we did the feedback linearization which is not based on the Lyapunov method at all it's just property of the system itself okay it basically gives you some kind of nonlinear state transformation which will make your system appear linear okay so that's really the idea so there are no other generic methods now it's more now it's more on what kind of problems you're trying to solve so adaptive control is one such problem in a like a scenario in nonlinear control yeah which occurs very commonly in nonlinear control and what is the scenario the scenario is that you have unknown parameters in the system yeah these unknown parameters could be mass inertia and things like that of course more recently some of you might be aware there is this neural networks and deep learning because of you know very good computational facilities now has become very very popular so all neural network and deep learning is doing exactly this identifying parameters okay so what it does is a in typical adaptive control the way we teach it we are just trying to learn some constant parameters of the system okay when we are working with neural nets and deep learning algorithms you are trying to identify functions not trying to identify points and parameters but trying to identify functions but there is a very very nice classical result which says that any function can be linearly parameterized in terms of the standard radial like basis functions yeah this could be radial basis functions or activation functions and things like that okay so so that is what neural network does it basically thinks of you know functions as a linear combination of some standard basis functions right and then all you have to identify is again some constant parameters alright so you are back to an adaptive control type problem okay so you can even use the adaptive control framework in learning yeah which is actually sort of well understood yeah so anyway so so applications of adaptive control are significant even in even before we were doing learning and stuff even when there are basic parameters or system that are unknown you know like mass inertia these are not easy to quantify especially when you talk big systems like spacecraft aircraft yeah or where you are losing fuel or there is some damage to say your propellers lot of unknowns or if there is a sensing error yeah so these all factor in as unknowns unknown parameters we again deal with constants here yeah but this has still a lot of utility yeah so these are scenarios where you cannot adequately model the system like if you if you talk about you know thousand kg or you know five thousand kg spacecraft you cannot really you know do rotational testing and all that to get some inertia values and all so whatever you have is a guess so it's better to then use something like an adaptive control okay alright so before we even do any adaptive control we need to look a little bit at some key results that we use very commonly in adaptive control of course we use the stability theorems that you already know but we also use a little few additional results yeah these are very powerful and so we need to state them and sort of look at how they are used yeah first so the first is there are few lemmas the first one is basically this lemma 1.1 which says that if you have a function f which is bounded below and not increasing okay so what is it it's bounded below and not increasing meaning if you have a function like this say there is a bound below and it's not increasing so it's like this it could be constant it could be going down constant going down constant going down yeah can never go up yeah this is the kind of function we're talking about it has a lower bound and it is non-increasing okay it's then this lemma says that such functions have a finite limit as T goes to infinity okay so limit as T goes to infinity f of T is some finite limit okay the limit exists and is finite okay so this is a rather key result that we constantly invoke in what we call signal chasing analysis this also something we look at of course there is this exercise which says what is this finite limit yeah I will leave it to you because it says there is a finite limit the question is what might this finite limit be yeah anyway so I'll leave that okay alright so the second lemma basically says it sort of gives you a result that has you to evaluate uniform continuity of a function if you don't know what is uniform continuity of a function please go read it up continuity is pretty simple you already know yeah again there are epsilon delta definitions for continuity similarly uniform continuity okay basically continuity does not depend on the point you are evaluating that is what is called uniform continuity in general yeah typically when you say a function is continuous you say condense at a point uniform continuity there is no continuity at a point it's wherever yeah okay but still if you are not clear you should look at the definition of uniform continuity all I am giving you is a sufficiency condition to verify uniform quantity what is the sufficiency condition if the derivative is L infinity okay and if you remember I told you L infinity is identical to boundedness any function L infinity implies the function is bounded exactly the same things okay so basically if your if your derivative of your function is in fact bounded then f is uniformly continuous okay this is an easy sufficiency check for uniform continuity yeah otherwise you have to check with the definition which is not easy typically typically hard yeah so simple examples you can see I mean because I know that f dot has to be bounded I know that sine t is uniformly continuous right on the other hand if I take let's see sine t squared is it uniformly continuous sine t squared yes but why how sine t what is the derivative of sine t squared this is bounded no not bounded yeah so you can't say anything about uniform continuity because this is only a sufficiency condition it doesn't say if it is not satisfying the boundedness what happens but this is not a uniformly continuous function sine t square is not uniformly continuous continuity depends on the t so when I say the continuity depends on the point it doesn't mean that it will become discontinuous at some point okay this comes from the epsilon-delta definition continuity says that if you are given an epsilon there exists a delta if so that if the argument is delta away from a point then the function is epsilon away from the point okay now that epsilon can depend on time sorry the delta can depend on t epsilon cannot depend on anything in uniform continuity the delta does not depend on t okay anyway go look go back and look at the definition of uniform continuity this is basically just a test yeah sine t square does not satisfy the test the sine t does satisfies the test sine t square no okay all right great right right yeah so anyway I mean I mean there is also an simple example here if you take if you take x t then you take the two vector norm then it's one yeah so this is just talking about boundedness so x is basically a bounded it just says that x is in fact in fact not just x x infinity yeah the infinity norm of the signal is 1 right again this is something we've already covered just how to compute the infinity norm infinity norm is just basically soup over time of this guy yeah so x infinity is supremum over time this right of any vector norm so here we take the two norm yeah this is basically just saying that it's a bounded signal it's fine it's basically just that's it's this is not talking about this result or anything no it's just saying that x is a bounded signal and therefore it is L infinity yeah any bounded signal is L infinity okay great so of course I have also given you this exercise define uniform continuity just so that you read it yeah and give the epsilon delta definition yeah not some arbitrary definition we need the epsilon delta definition okay now unfortunately in these notes everywhere this is wrong it is barbell art's lemma why we have made this blunder here it's become Barbara at slimmer it's not a rat so Barbara lots lemma there is no rat involved here all right okay this is Barbara slimmer all right so so why we talked about these results is because we wanted to reach up to Baba lads lemma this is a sort of equivalence or extension even of LaSalle krasov ski LaSalle theorem in some ways okay if you remember the LaSalle invariance is talking about convergence to a compact set okay but the krasov ski LaSalle barber sheen krasov ski LaSalle right theorem was talking about convergence to the origin okay when when the V dot is negative semi-definite only okay but if you remember everything we did in the class or in the LaSalle invariance and I required that the system be time invariant autonomous system and we were always dealing with autonomous systems here the Baba lads lemma is going to state an equivalent result but not necessarily for autonomous systems okay it's got no can no you don't have to have an autonomous system okay but again this is only generalizing the barber sheen krasov ski LaSalle theorem okay not the LaSalle invariance LaSalle invariance is completely different and way more general because it is talking about convergence to a compact set yeah Baba lads lemma does not do anything like that so what is this Baba lads lemma saying it says it's a convergence result like I said it says that if you have a function can be scalar or vector value doesn't matter of course it's a function of time therefore it goes from I have said R alright such that the signal is integrable what is integrable mean? Integrable means that this integral exists and is finite okay so if you if you integrate it from 0 to infinity then it exists and is finite okay further suppose f is uniformly continuous okay then limit as T goes to infinity f of T is 0 so this is the convergence as you can see you are talking about convergence okay and and we have also of course later on you will see how we use it for states because this is just talking about a function right but if you remember the state is also a function of time once you solve it once you solve the equations the differential equations it's a function of time and also initial conditions okay but still a function of time right once you fix the initial conditions also alright so this is a nice convergence result it says that if the function is integrable and uniformly continuous then the function goes to 0 as T goes to infinity okay there is also a of course there is a nice note which says that in case of vector valued functions the integral has to be satisfied component wise so basically it means that component wise you want the the integral to have a finite limit okay that's it so there is a simpler version or a corollary yeah what is the corollary the corollary is basically that if the function is L infinity and LP for some P which is not infinity of course and further f dot is an infinity then limit as T goes to infinity f of T is 0 so this is a corollary of the previous result this is a corollary of the previous result just note why this is a corollary the first thing f dot being in L infinity already implies that the function is uniformly continuous correct so that's what is the second condition here already I get one condition now this other condition it looks like an integrability condition right why because first you are saying function is L infinity which means it's bounded so leave that aside but if the function is LP what do you have what does it mean for a function to be LP it absolutely the the P norm is integrable the P signal norm what is the P signal norm the P signal norm is this is what we define the P signal norm right now if I say this is integrable it is basically as good as saying that this is I mean to the power 1 pi P doesn't matter right if this is going to infinity the to the power 1 by P is also infinity and vice versa okay so when I say integrable when I say that function is LP I know that this is less than infinity all right this already looks similar to this guy where there is no power of course right there is no power involved here but very similar looking condition right so therefore this is in fact a corollary that's what I ask you to prove I am going to cut this and I am going to say prove that limit as T goes to infinity yeah basically I am saying if you have all these conditions I want you to prove that the function goes to 0 yeah basically I want you to use these conditions to go back to the barbell arts lemma original conditions and therefore you have F going to 0 as T goes to infinity okay alright so that's why so basically proving that this is a corollary okay so this is the barbell arts lemma so the next step is to basically see how we can use the barbell arts lemma yeah this is significantly simpler than all your feedback linearization materials so you will follow this rather easy okay.