Nonlinear Control Design

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Week 10 : Lecture 58 : Feedback Linearization: Part 9

So, what we have shown is that we started with a span of these vector fields which was some distribution. We said that it is involutive by involutivity we were able to show that in the first step that you can transform it to some new vector fields which commute that is the lie brackets are 0. The new set of vector fields have 0 lie brackets then you are able to show that the original vector fields can be transformed by a state coordinate transformation to e 1 to e k and finally, that you can get annihilator vector fields h j in the original coordinates. In the new coordinates anyway you have the g j's but you can get the h j's in the original coordinates also which are basically the annihilators right and these are exactly going to be the addition the h j's are going to be the additional coordinates that we after relative degree r we have to construct some n minus r coordinates these are exactly those ok. So, what are we left with the converse I said that involutivity and complete integrability are equivalent. So, we are left with just proving the converse ok and that is what is this guy ok all this does is proves the opposite side of things yeah anyway.

So, we are we do not have to look at this but the converse proof is pretty straight forward once you have complete integrability proving invertibility is not difficult because complete integrability not just tells you the existence of these h functions it also gives you that these d h corresponding to these h functions are linearly independent ok. So, you have enough to claim involutivity in fact alright. So, what is where we wanted to focus was what is the implication on feedback linearization ok if you remember we started with a system like this yeah single input system yeah. So, we had only an f and a g alright.

Now, suppose I claim that this distribution which I have constructed in the rather interesting way that is its span of g add f g all the way to add f k minus 1 g ok is non-singular and involutive suppose this happens ok. I mean we will see why or why we have constructed like this, but these are my f 1 to f k I do not need actually separate f 1 to f k this is my f 1 f 2 all the way to f k ok I am just constructing it out of the ingredients I have here fine. Suppose this happens I know by Frobenius theorem that delta is now an integrable distribution alright what does that mean it means the existence of these h functions right how many n minus k exactly n minus k and with what condition that this happens I just wrote f 1 to f k this is just f 1 to f k add f 0 g add f 1 g all the way to add of k minus 1 g yeah that is f 1 to f k and h j is all of these ok. Does this condition look similar? Same like lemma 0.1 0.

2 this is all this is the kind of stuff we had ok what is this I am expanding this let me expand

this thing now I am expanding not just this one, but all of them there are many how many are there how many conditions are n times k conditions right this is because there is k here and n minus sorry let us see n minus k times k conditions right I believe this is n minus k times k conditions right because there are that many I am going to expand them how in this way ok because I just I am just writing the lie brackets I am not doing anything just writing the lie brackets etcetera this is del h 1 del x del h 2 del x all the way del h n minus k del x what is the dimension of this guy n minus k rows and n columns yes because each of them is a row vector each is a row vector each is a row vector and multiplied by this vector field that is I write all of them add f 0 g add f 1 g all the way to add of k minus 1 g what is the dimension of this n cross k why because this is a vector field n dimensional n dimensional and there are k such elements so n cross k what is the product how what is the dimensional product n minus k cross k right that is the that has many conditions that I have right. So what I have done I have taken this one small condition and written it in this matrix form do you remember this matrix we had no we had right in 0.1 0.2 lemma these are the this is the sort of matrix we have been looking at right very familiar to us ok. So again different looking right but very similar I mean I mean just this thing inside as d h 1 d h 2 and all that right other than that this guy is exactly the same is r is just replaced by k alright otherwise exactly the same what am I showing now there I use it for a different purpose but you see I am going back and forth to the same sort of expression that is the important thing to remember alright great.

But now integrability further tells me something that these guys are linearly independent right that is that is the meaning of complete integrability and fobini's theorem gave me complete integrability for free by involutivity ok excellent. So what does it mean d h 1 d h 2 all the way to d h n minus k are linearly independent what does it mean this thing has to be rank n minus k yeah because there each row is linearly independent right. So of course this entire matrix cannot have rank more than n minus k this is rank n minus k ok. So obviously what do we have now this is this guy is of course 0 ok no problem. So what we are now saying is that these h 1 to h n minus k become my new coordinates so relative degree is actually not n minus k relative degree is relative degree is k right not n minus k n n minus k are these additional coordinates that we construct in which the control will not appear right.

Obviously control will not appear because if I take derivative of h 1 of any h i for that matter wait a second yeah h i dot I get l f h i plus l g h i u yeah so 0 ok. So there is no control in this ok and so on and so forth yeah you basically get these new additional coordinates like I told you right these h are what become your additional coordinates when you are looking at partial feedback in addition ok. Now I do not have anything here I have something else there but what does I mean it is nice to also wonder what is involutivity of this vector field mean of this delta mean ok. What does the involutivity of these guys mean ok. Involutivity means that if I take d bracket between any two of these any two then that has to be in the span ok.

So suppose I actually see try to look at that ok try to see if it really happens right. So if I take say l e bracket of g and add f g what is this is this anything that we know. So what is add f g by the way add f g is yeah add basically gives the successive l e brackets right ok. So what does this give me what do you think happens here. So we have to do the entire computation right we have to do the little bit of the computation this is a little bit complicated yeah let us see if we can let us give it a shot even I have not tried it.

So this is d add f g g minus d g add f g yeah ok. Let us see does this help us in some way can I simplify this further or this is it apparently not yeah yeah. So I was thinking you have to simplify this further but no does not look like I can simplify this further ok. So I will see I am just going to make that remark cannot I was thinking if there is some nice expression or simpler expression that will come out of it does not look like it not on the face of it unless one of you can work it out and tell me but I do not think it is turning out to be any further simpler yeah. What this condition is is that this is essentially the relative degree condition.

See what you were doing is what we were doing is we were basically taking the output a particular output all right. Remember the entire feedback generation that we were doing until now was relying completely on a particular output ok. What if you do not know what is that output how do you verify if you have any relative degree and feedback generation and things like that yeah then you have to use this kind of a method where now you notice this is not dependent on any output this condition. So what you have to do is you have to if you have a single input system here like this and you have no idea what output you can choose that will give you some feedback generation because otherwise you can do bunch of trials and errors right. You will have to check these vector fields for this distribution yeah and what will you have to do you have to verify that this distribution is non-singular which means what that each of these are linearly independent for all arguments p at every point they give you linear independent vectors ok.

That is what it means for this to be non-singular ok and involutive you will have to actually check that they are involutive ok. Unfortunately it is not obvious that it is still giving you any output yeah at the end of this exercise if you see it is not very easily evident what is the output that you can use ok. But when we are doing the full state feedback generation you will see that it becomes very easy to find the output using this method also. So basically this gives you a method of like sort of finding that output that will give you a feedback linearized system yeah. Here not so much I will also have to see how to get the output here.

One thing that is sort of evident here is that these coordinates that I got right if I keep taking successive derivatives alright so I take the first derivative I get no control alright no problem. I take a second derivative I will get what L F square H I right I will get an L F square H I no control again ok. But similarly if I take K minus one th derivative I will get L F K minus one H I right and still no control ok until this point here yeah when I take the K th derivative right so this is this is basically add F zero in one so yeah in the K th derivative of H I I will get add F K minus one G H I yes this is also I believe zero right correct because right by this this guy is also zero right. But if I take H I K plus one ok then the control will show up ok but that is still vague yeah it is not evident that that is a good choice or not ok so I am just commenting on the fact that this H I themselves could also lead to feedback linearization yeah the output that will give you feedback linearization ok not just as the extra out extra states ok one way to look at them is that they are the extra states right but they could also lead you to the output that will give you feedback linearization ok so it is not very evident here because you need additional conditions you need to know that L add F K G H I is actually non-zero that condition is not being you know posed anywhere here yeah you need that the higher derivatives actually have non-zero input so that does that happen or not is not very clear in this case ok so anyway so this is the implication of Frobenius theorem in partial feedback linearization basically it comes into play when you don't have an output which you can keep taking derivatives of and try to find the control yeah you may not have a suitable output in those cases you can actually directly use the vector fields of the system itself and the Frobenius theorem to see what is the relative degree of the system it is not very obvious in the case of partial linearization that how to find this output ok this will give you the relative degree but how to find the output is still a little bit of a jugglery at this stage ok now if you go to the full state linearization then something more can be done ok that's what we want to look at now alright so we are again back to the notion of integrable distributions don't worry about it this is a repetition I already told you that if you have a distribution which is a span of some vector fields right then if you think of annihilator theory you are looking at some y's which are annihilating each of these that's essentially what we have the DH for ok so that's what it's mentioned here in a just a different notation please don't worry about it again you say that the distribution is completely integrable if it is annihilated by these n minus R lambdas ok this is just different notation yeah going from H to lambda yeah of course you also know what is involutivity I am not going to actually stress on it again so we want to look at now the conditions under which you can completely feedback linearize ok we've already looked at the condition by the way there is nothing very big about it we've already looked at partial feedback linearization so when k becomes you know when this k becomes n you have complete feedback linearization alright that's it right because you've already looked at the case when for arbitrary k so if you k becomes n it is completely feedback linearizable and that's what we are writing here yeah this is again for this particular system x dot is ok doesn't mention here but this is again for the system assumed is still x dot is f of x yeah for this system actually I apologize this should be this section for this section this is the case yeah alright so what have we done we are talking about some necessary and sufficient conditions right and we are constructing this matrix ok g add of g add of n minus 2 add of n minus 1 ok we want this to be rank n what is this matrix same thing that we constructed right just taking k equal to n right same deal what because what does it do if all these are linearly independent then you get a non-singular you know a non-singular distribution alright so what that's the first requirement that this has rank n yeah he is talking about a particular you know point x 0 ok but typically you will want it for all x 0 ok alright ok great so this this is rank n is the same condition that we saw before because this is what gives you the distribution Delta ok right next now we are looking at the distribution here well fine in this particular case the one of them is removed from the distribution ok in this particular case the distribution is removing one term ok this is for a particular reason for a very specific reason we look at it subsequently alright so the distribution is not spanned by g add of g all the way to add of n minus 1 g but we are going only until add of n minus 2 g ok alright so we are asking for the distribution to be involutive just like before ok so same conditions we are creating a distribution based on the vector fields in the system which is f and g right using the f and g we have constructed this you know distribution and we want it to be non-singular and involutive alright and yeah that's it we want it to be non-singular involutive and of course we've also said that for non singularity we have sort of added this additional term here ok we have added this additional term alright here the h function or the lambda function is used in a slightly different way ok that's what I want to highlight now I hope this is clear can you do you see the identity between this and what we did in the partial linearization are you completely lost you see what we are doing here at least I hope you can see that these expressions are very similar ok otherwise we are just trying to apply the Frobenius theorem which is saying that if your distribution is involutive it is completely integrable and complete integrability means I get these functions h and I want to do something with those functions h ok you don't know what what we want to use them as but we want to use those functions h because they have some you know these nice properties which is basically something like this annihilator type properties ok so that's really the whole plan alright so let's see even if we forget what we have looked at earlier let's look at this in a separate context so we have taken these vector fields ok and again we will also try to look at some examples of course and we are saying that these vector fields are independent and therefore rank n and further now we construct a distribution out of all of them but one so the last one I have removed ok so now this has only n minus 1 vector fields so what will be the dimension of this distribution what would be the dimension of this distribution if it's a non-singular distribution because I already said G add of G these are all linearly independent yeah so what would be the dimension of this distribution how many vectors does it have n minus 1 ok and what is the size of each vector n so this is an n by n minus 1 if you look at it in matrix form is n by n minus 1 right so what what can be what is the rank of this distribution then n minus 1 cannot be more than that n is larger ok so the idea is that rank of D is n minus 1 this is why I have the non-singularity ok I have already assumed involutivity ok so I can immediately invoke the Frobenius theorem and say that it is completely that this distribution is now completely integrable because it is non-singular and let us check that ok.