Nonlinear Control Design

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So, that is what I was saying that this result is very powerful because it really gives an easy check. If I have k vector fields in the distribution, I just have to check their combinations. Yeah, so k C2 and you are done. Okay, check that their combinations belong to the distribution and you are good to go. Okay, proof, of course whenever we do something we have to prove. It is an if and only if, so you have to prove both sides.

First suppose if you, that you have involutivity. If you have involutivity then this is obvious, right, because involutivity means that if F1 to Fk belong to delta then the lie bracket have to be in delta. So this is obvious by in, you know, that if you are starting with involutivity obviously this will happen, no problem. The bigger question is if you start with this, that only their combinations are in delta, then can you claim that the distribution is involutive as per this definition.

Okay, how do we go about it? We take some two vector fields, okay, G1 and G2 in delta. Now the important thing is because G1 and G2 are in delta and that happens at all points, you should be able to write them as a linear combination of the Fi's. That is the only way G1 and G2 can belong to delta, right, because they have to be the linear combination of the Fi's, right. And of course these Fi ij's are smooth functions, yeah. Smoothness is coming out of the fact that Fi fj's are also smooth, yeah, and G1 G2 are also smooth, okay.

Therefore this coefficients have to be smooth, okay. So everything is smooth so the coefficients also have to be smooth, okay. For simplicity if you assume k equal to 2 that is you have only two vector fields here, F1 and F2 forming the distribution just for so that the proof looks easy, yeah. Then what do you have? You have that the lie bracket of G1 G2 is actually this guy, lie bracket of this, okay, because I have written G1 as Fi 11 F1 plus Fi 12 F2 and G2 as Fi 21 F1 plus Fi 22 F2, right, because G1 G2 belong to the distribution, right. Now I want to compute this lie bracket, okay.

Now there is a problem with this computation, okay. I want you to actually this is another exercise. I need you to complete this computation. This computation is pretty straightforward. It is just doing this with some coefficients, Fi 1i Fi plus Fi 2j Fj, remember Fi 1i and Fi 2j are depending on x, they are functions, they are not just, yeah, they are not just constants or anything, okay, alright.

So I want you to prove this computation anyway, but let us look at the lie bracket of G1 and

G2 as per this expression, okay. It is known that the lie brackets are distributed. I have not talked about this, yeah, but it is easy to prove that they distribute, that is I can break this open. How? In all these four terms, yeah, this works like a nice distributive bracket, okay. So from G1 G2 I will get Fi 11 Fi 21 F1 F1 first term, then I will get Fi 11 Fi 22 F1 F2 second, Fi 12 Fi 21 F2 F1 and Fi 12 Fi 22 F2 F2.

Now it should be obvious to you that the lie bracket of a vector field with itself is 0, just yeah, 0, okay. So these two are gone, okay, these two are gone. So what am I left with? I am left with these two, okay, and these are actually the same term, I can actually flip the sign. So also this is another thing that is true, lie bracket is anti-symmetric. So F1 F2 lie bracket is negative of F2 F1 lie bracket, yeah.

So I can flip the sign. So what am I left with? I am left with something rather simple, right. This is my G1 G2, okay. Now what have I assumed? That F1 F2 belongs to delta, yes, yes, which means what? G1 G2 also belongs to delta. Why? Because it is just a scalar multiple of F1 F2.

Yeah? No? What do I have to prove for invulnerability? Take arbitrary two vectors, vector fields in the distribution and prove that the lie bracket is in the distribution, yes. What does it mean to be in the distribution? That they are linear combinations of the forming vector fields, okay. I have assumed that there are only two vector fields, forming vector fields, yeah, or generator vector fields which is F1 and F2. So G1 and G2 if they are in delta means G1 can be written as phi 11 F1 plus phi 12 F2 and G2 can be written as phi 21 F1 and phi 22 F2. This phi 11, phi 12, phi 21, phi 22 are functions of x but scalars, they are not changing the vector direction.

This is in the direction of F1, this is in the direction of F2. It is like multiplying a scalar and a vector and just that here it is not just scalars, it is a scalar function. Similarly, it is not just vectors, it is a vector field. That is it, okay. So phi 11 F1 is in the direction of F1, phi 12 F2 is in the direction of F2.

Similarly, phi 21 F1 is in the direction of F1 and phi 22 F2 is in the direction of F2, okay. The important thing to remember about lie brackets is that they distribute nicely. That is what I have used. I have just broken open the bracket just like you break open any product, works exactly like this, yeah. So I get four terms.

Now out of these two terms have same, you know, F1, F1 and F2, F2, okay. Just because the lie bracket is anti-symmetric, this structure, this is 0. If you put, if G and F are the same, this is actually the 0, you know, becomes the 0 vector field. So this guy is 0, this guy is 0. I am left with F1 F2 and F2 F1.

And so that is negative, so F1 F2 is negative of F2 F1 again by anti-symmetry. So all I have left is this entire guy, okay. And this is just a scalar, yeah, I mean scalar function but still a scalar, not changing the direction of the vector, yeah. And I have already assumed that F1 F2 belongs to the distribution. So I am done, G1 G2 belongs to the distribution.

Now if I had assumed instead of two Fs, there were many more K generating Fs or say three, what would happen? I will get three, 511, 512, 513, 521, 522, 533. But I will still get combinations of these and repetitions, combinations repetitions. And the combinations are already assumed to be in the distribution. So the combinations are dying. So therefore I will always remain in that, okay.

So straight forward actually, yeah, you do not have to worry too much. Yes. Good point. I was also wondering, one should always ask oneself, are we using all the assumptions? A non-singular distribution, if delta is a non-singular distribution, then involutivity if and only if, okay, okay, okay, okay. You know what, so what you are asking is that where do we use the non-singularity of the distribution, alright.

Now you know this is where I think the problem will happen, this place. I think this equality is no longer writable is what I am wondering or am I wrong? I think this equality is where I will have start having trouble. Yeah, I have to think about it more carefully but I believe this equality you will not be able to write anymore because if your distribution is changing rank, the span is changing rank. Now the span definitely has to hold the vector because we are assuming that g i is in delta. So obviously the span has to hold the vector but my feeling is you will not be able to do it with smooth functions here.

Yeah, if you suddenly want to jump from one point p to another point p prime, you go from a three-dimensional distribution to a two-dimensional distribution, alright, because it is singular, then this phi will undergo a rather drastic change. I do not think you will be able to retain smooth phi's anymore. Again I have to think about this more carefully but this is where I think things will go wrong. This assignment will not lead to smooth functions phi but that is a good point.

I will try to hunt it up. That is also the next question I was asking myself. Alright, where are we using the smoothness of the functions phi? See when I look at this without a smooth phi, this is poorly defined. Fine, I wrote this as this split and all this, alright, but if I try to actually compute this guy by this formula, you can see I am starting to take partials of the phi's and here. So this is poorly defined. So even writing like this is not okay because I mean eventually all the bracket operations are, these are all derivatives of some kind or the other.

So I am frequently taking derivatives. I will definitely mess up. I mean this is not a good, I mean not a well-defined object anymore. So it almost certainly, I believe what I am saying is right that non-singularity will result in these phi's becoming non-smooth because you cannot just jump rank and expect that everything will turn out to remain smooth. This will definitely create some trouble.

You are suddenly projecting to a plane. Say you are in three dimensions, suddenly you are projecting to a plane or in five dimensions to a four dimensional hyper plane or a three dimensional hyper plane that will not retain good values of the phi. I mean, yeah, I mean as of now whatever I am saying is imagination. It is nicer to if we can construct and sort of evaluate. But I believe that is what will go wrong.

I will check anyway if I can find some fun examples. So here I am just giving some examples of this normal form business. This is again going back to the previous topic but anyway let us look at this. So this is sort of the system and I want to put this in normal form. You can see it is already non-linear and messy.

I am using y as x2. I am using y output as x2. I want to get this to normal form. How do I do this? I simply start taking derivatives first to get the relative degree because anyway the y and its derivatives become my states, my linear states. So the first state is x2 itself. Then I take a derivative and I get to x2 dot which is x3 and then I take another derivative y double dot which is x3 dot which gives me the control.

So this is a relative degree what system? What is the relative degree of this system? Two. Yeah, it is two. I have also written it here. I took two derivatives of the output. Whatever output I was given, I took two derivatives, I reached the control.

So obviously my first two states become this guy. That is y and its derivative. Those become my first two states or the last two states. Then I need to find the phi state which will make it a diffeomorphism.

Suppose I choose this guy x1. This is a problem because the derivative of x1 contains the control. So I don't like it so much because it is not going to give me the normal form. Now what do I need for the normal form? If you remember, I need L, yeah, I have actually written it here. I need Lg phi equal to zero.

So that there is no control term appearing. So what is Lg phi? Lg phi is basically partial of phi with multiplied by g. What is g? g is basically whatever is multiplying the control. So that is 2 plus x3, 1 plus x3 is whatever. 2 plus x3 square, 1 plus x3 square, 0 and 1.

This guy. So if I compute this product, I get this. I can actually break this open to get 1 plus x3 square this, del phi del x1, del phi del x3. Now I want this to be zero. I want to use this to motivate my new state.

So I have of course chosen it like this. Yeah, pretty scary looking actually. How do you think I came up with this? So this is my phi. This is what I chose as phi. How do you think I came up with this? I did some guesswork. So if you look at this guy, yeah, this looks like tan inverse, derivative of tan inverse x3.

This looks like derivative of tan inverse x3. So if I keep x1 as linear, I get del phi del x1 as 1 and then I get del phi del x3 as 1 over 1 plus x3 square. Yeah, 1 over 1 plus x3 square and then this guy is again just multiplying del phi del x1 which is 1. So these two will cancel out and I'm left with, my apologies, did I get this right? Yeah, yeah and this and part of this will cancel out. This is basically del phi del x3 is going to give me minus 1 minus 1 over 1 plus x3 square.

Okay. So I did a little bit of guesswork. How would you do it if I wanted to guess a phi starting from this equation? Is there any other smarter way? I thought it just bunch of guesswork. I just looked at what I can cancel. Is there any nicer way that you can think of to sort of arrive at this? No? No? No? Because individually I can't say that individually these are 0 or something like that. It's not a quadratic or anything. So I can't say that individually these are 0 or some such.

One thing that's obvious to me is that I don't have the, this entire equation has only x1 and x3. So I don't need the phi to depend on x2. That much is obvious to me. My phi doesn't have to depend on x2 because it is playing no role in this entire equation.

So phi is only a function of x1 and x3 is all I know. Yeah I think to me it seems this is what works fine. Okay. And anyway if I choose this funny looking coordinate I get z1 dot as this, z2 dot as z3 and z3 dot as this.

z1 z3 plus u. Yeah. Oh no it won't work. x1 minus tan inverse x3 is it? So del phi del x1 will be 1 and del phi del x3 will be minus 1 over 1 plus x3 squared. So this will be 1, 1 over 1 plus x3 squared and this will be 1 over 1 plus x3 squared. So 1 remains no? So I am trying to cancel the 1 also. That's why I put the minus x3 here. I must have tried with that only but then I added the minus x3 because I have to cancel the 1 also.

Okay. That's it. Yeah it's just little bit of hit and trial. Okay. But this is what you get as the dynamics. You still get nice enough zero dynamics by the way if you see.

Yeah. Because if the linear part goes to zero this guy is zero and this is again an exponentially stable system. Okay. So again started with something complicated and very nonlinear. Not evident how I would design a control based on any method that I know.

Yeah. But I still end up with something rather nice. Okay. Oh this one I guess it's already in your notes. Why is this exercise then? I see. Apparently it's not exercise. I already solved it.

So it's already in your notes that I have uploaded. So I should probably just not say this is an exercise.

Solved. Yeah. Solved. Yeah. Because you can see that it's here I mean I've actually done it. Anyway you can take a look at this. Yeah. You can take a look at this in your leisure time.

This whatever expression you get here. Yeah. I believe this expression has a problem. This expression has a problem. So anyway. So I got to this expression.

I'm not sure why I'm saying this is specialized to k equal to 2. Doesn't seem like this is k equal to 2 is necessary here. Yeah it seems it's specialized to k equal to 2. Yeah because otherwise there would be summations and stuff.

That's all. Alright. Okay. Anyway this is something you can just look at it on your own. Alright. So anyway next time we will continue with our discussion on Frobenius theorem. Alright. We will stop there. Thank you. Thank you.