Nonlinear Control Design

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Week 10 : Lecture 55 : Feedback Linearization: Part 6

Great, so we know enough about the zero dynamics. So the idea is this, that if the zero dynamics is stable when the linear part is zero and the linear part is also stable, then the combination is stable. I mean when I say stable, I mean asymptotically stable. So that is the general case. As like I said, cascaded systems literature pretty strong, actually talks about such results only if you remember. I mean we spoke of a very specific case where we had a stable system to begin with and then it was being driven by a passive system.

But you know that what is the purpose of passivity, because of passivity you got essentially some kind of nice stability, right, because once you have V dot is U transpose Y, then you construct the U as some phi Y and then you have actually stability, right, you have something nice. So this is also something similar. You are saying that you have a system, this kind of a system, the nonlinear system which is sort of you can think of it as being driven by this linear part and if this linear part is going to zero, fast enough or whatever, going to zero, then you have asymptotic stability for the combination. What I have not written, I have not written this but this is something I wanted you guys to look at, okay.

Exercise. Yeah. Yeah, so I want you to complete this proof. Yeah, because what am I saying? I am saying that the system is locally asymptotically stable or asymptotically stable whatever with psi equal to zero, so you have a V eta, right, it's given to you such that partial of V with respect to eta times Q zero eta is negative definite, okay, and then you have this to be Hurwitz. So therefore with this also you have some psi transpose P psi as your possible candidate Lyapunov function from and the P you get from this Lyapunov equation, nice Lyapunov equation.

So what I want to do is to investigate stability of the entire system with this new candidate Lyapunov function, okay, which is basically just the sum of this and this, okay. I want to see what or how you can do it or what you can do, yeah. You can make assumptions if you want, alright. You can make some assumptions if you want, alright, that's fine. Yeah, if you want to make some assumptions you can make but see if you can get something interesting, yeah.

So again the motivation is cascaded systems, things like this, motivation is cascaded systems that look like this, alright, so great. So that's the idea, we can do nice, get nice stability, right, that's the whole point. I don't know, let's see, that's before going to Frobenius, let's go back to our example, right. So this was our DC motor, the transform dynamics, right, the Z dynamics, transform dynamics. Here what was the xi? xi is this guy, the first two and this is the eta, okay.

So you see this has this nice structure, right. If xi is going to 0 or if xi is equal to 0 means what? Z1, Z2 is 0, right. Notice Z1 is x3, Z2 is theta x1, x2. If Z1, Z2 is 0 means x3 is 0 and x1 or x2 is 0, right, which means most importantly that this term is 0, right, okay. And when this term is 0 what can I say about this dynamics? Exponentially stable, yeah, Z3 dot is minus BZ3.

So this is the minimum phase system, yeah, the 0 dynamics is exponentially stable, more than asymptotically stable, it's exponentially stable, alright. So this is a rather nice system, right. Why? Well, all I have to do is check when xi is 0, when xi is 0 what happens to the eta dynamics, so when xi is 0 these guys are 0 which means this guy is 0, this is gone. So what am I left with? Z3 dot is minus BZ3, it's exponentially stable, it's very nice 0 dynamics. So we can actually use these results.

All I have to do is design a U so that this system becomes exponentially stable. That's also easy because I know that this is non-zero, this is non-zero, I mean that's the assumption. So I'll just divide by this theta x2, whatever, I mean I invert the theta x2 and I can give it some minus k1 z1 minus k2 z2, right. Then basically this becomes a spring mass damper, right, this whole thing will become a spring mass damper and then so that's also exponentially stable, right. And then output of that basically is just going into this guy, yeah.

So you can immediately apply these results to get a stabilizing feedback for the DC motor. As you can see we started with a relatively complicated non-linear dynamics, right. It was not like too straight forward, I don't know where it is. These notes are a bit difficult to read. So we started with relatively complicated dynamics, right.

I mean there is enough non-linearity here, yeah, for you to be able to immediately design a control right from here, right. If I'd asked you to design a control for this system directly it would not have been easy for you, yeah, would not have been easy for you. I mean even I can't see how I would do it because the control appears only here, right. So I can do nice things with this guy, sure, no problem. But what about these? Very non-linear terms.

I mean can't even wrap my head around how I would have designed a control. Back stepping, nothing will work here. I don't see what will work here. I mean backward back stepping something, I don't know. Like start from here and then go here and then go here or something messy like that.

But nothing is obvious to me. Passivity, definitely not obvious to me. No way, I don't see what is, I mean this how I will get passivity with this guy and this guy. I really don't know, yeah, no idea. Not obvious what would be a storage function for this, right.

Not at all obvious. So if I look at, started with this system which is a standard DC motor model, I don't know how to design a controller, okay. But with some whatever playing around like we did, we have this system where I now have a very good handle on how to do things. I don't even have to do design anything here. This is what it is. All I have to do is design a stabilizing controller here, yeah.

Very straightforward, right. I mean can't get easier than this. I don't see it. I'm not saying it's impossible. I'm not saying it's impossible.

See this is the output given to me. This is the input. You need to find a storage function such that V dot is less than equal to U transpose Y. See it's a fact that every system is passive or not, almost a fact that everything that you see around you is a passive system but you have to pick the right output and input.

I don't see it as of now. What is the output and what is the input, okay. See with standard mechanical systems what do people do? They take energy as the storage function and work with it. Here there is no obvious storage function available to me. So that's why I'm saying it's not obvious to me. And so these are the only methods I know, backstepping and passivity.

I don't know any other method. And to be honest there are not that many other generic methods either, okay. Of course there is the CLF based method. Then the question is can you find a CLF for this system? I mean is it obvious what is the CLF for this system? Again not so much to me.

Yeah, I don't know. Yeah. The first thing I would do obvious is to which is what we did I think with the XZ3. The Z3 was I think X3 minus K3. Yeah, Z3 was X2 minus K by B. That seems like an obvious transformation because that will make this equal to Z3 minus Z3, right.

This is quite obvious. That's one thing I would do that I would make this Z3 or whatever some other variable instead of X2 say some other variable instead of X2 and so the derivative remains the same but I get something nice here. But that messes this guy. The new variable will mess up with this one. So honestly not obvious even how to construct a control layer function.

Yeah. I mean I don't know. Again folks in electrical who do DC motor stabilization might be knowing what is a valid V for this but I don't know, okay. So of the top of my head I don't see how the methods that I have learnt apply here. Yeah. But you see feedback linearization gives a clean answer here.

Yeah. It's not even evident that this system is you know will give you get you to something nice like this. Okay. All right.

I mean not evident to me. All right. Okay. All right folks. So good. Good. So feedback linearization is relatively powerful.

Yeah. Because you are doing a state transformation by using some method it almost look like some bunch of magic portions or some incantations. Right. I did some LGH, LFH and bunch of LGLFH and LFLGH and whatever and add FH, add FGH and so on and so forth and I reach some fun place where I can get some nicer looking system.

Okay. So quite powerful actually. Yeah. Good logic to why it works but it's still quite powerful. All right. Great. So this I want you to verify as an exercise that what happens if I use this V to this combined system.

Let's see what you can get. Now we move on to a more involved theory topic. Also on feedback linearization this is where we start to answer the question when does such an output Y equal to HX exist. When do you have an output such that you can get linearization. In fact in this case you look at full state linearization not partial linearization. When can you actually when can you guarantee existence of such an output Y.

Okay. So those are the questions that are being answered here. This is why things get a wee bit involved. Now so this is what is the context of Frobenius theorem. Okay. Again I'm freely using notes that Vivek had used a while ago.

I have of course made some notes here and there. But we start with the notion of distributions. Okay what is a distribution. Distribution is a pretty straightforward idea.

You already seen vector fields. You have seen f and g which is f is the drift vector field g is the control vector field. Typically your system looks not like typical dynamics is. Yeah your typical dynamics is like this. You have multiple control not just one control. So you have a drift vector field and you have multiple control vector fields.

We keep calling them vector field because at every point it gives you a direction. Every point in the state space what does it do. It gives you a direction like this. What is drawn here. You take a point f1 gives you this direction say f2 gives this direction.

What is a distribution. Distribution is just a span of this. Is the vector space that is formed by this. Okay. So if you have some k vector fields. Then a distribution is basically assigned to each point this subspace.

This is the span. Okay so if you have two vector fields. Okay and you take the span of these f1 and f2. Then basically you can get any vector in this more or less in this plane. Because this is linearly independent vectors.

So you get anything in this plane. But of course if you think higher dimensions. You may not get everything in the plane and so on and so forth. Right. So this is what it is. Its distribution is just a way of representing a span of these vector fields.

Okay so which means that any element in the distribution. Yeah so p is of course you need a base point. Because these vector fields are being evaluated at a particular point. Right. That is why it is called a vector field and not a vector.

Yeah because it is being evaluated at every point. It is a function. Yeah so if you plug in this particular point p. Then at that point you take the span of all of this. That is called the distribution delta of p. And it is also notationally accurate to write delta as span of f1 to fk.

How do you find the dimension? Just take the rank. Right. But the point is you can see that the dimension depends on p. Right. As you can see because I plugged in a point p.

Right. The way we have been doing dimension. You take the vector span and then you take the rank of that. Right. Here each of these depend on point p.

Therefore the rank also depends on the point p. Okay. That can be a problem. It can vary with p. But that is not what we look at. We usually look at distributions which have constant rank. Yeah which means dimension remains the same for all your entire space of interest.

Yeah. p in x. And such distributions are called non-singular distributions. Okay. Such distributions are called non-singular distributions. So the concept is pretty simple.

Instead of vectors you have vector fields. So you have to plug in a point to get a vector at that point. Right. And then the distribution is just the span of all these vectors at that point. Right. The rank is basically just taking the rank of those vectors at that point.

If the rank is independent of the point you plugged in. Then it is a non-singular distribution. Okay. We will only work with non-singular distributions. Alright. And why? What is the purpose of all this mess? If you forget the f, let us suppose there is no f, then you have a system which looks like this.

Alright. And you see at every instant by choosing the control where can you move? At every instant what is the direction of your velocity? It is in the span of fi. Right. With whatever control you choose, instantaneously where can you move? You can only move in the span of fi.

Right. You cannot move anywhere else. That is the whole relevance of all this. Yeah.

Talking about distribution. Yeah. Because this is dictating the velocity. So instantaneously you can only move in that direction. Cannot move in any arbitrary directions.

Okay. So if I create a vector field for you, which is well known. Right. You have a car. It is a standard car. You cannot move instantaneously.

Your velocity cannot be orthogonal to the wheelbase. Look. If your car is pointing forward, all wheels are forward, you cannot instantaneously move this way. Okay. This is a constraint of the system.

Instantaneously. Remember, I said instantaneously. Right. So I cannot move sideways. Okay. Because that system prevents me. So if you write the dynamics carefully and you compute all this fi's, you will find the distribution does not contain this orthogonal vector.

Okay. Anyway. That is the motivation for talking about distributions and non-singular distributions. We say that a vector field belongs to a distribution f and written as f delta if f of p belongs to delta of delta at p for all p. Okay. A vector field is said to belong to a distribution if f of p is in delta of p for all p.

Simple. Further, a distribution is said to be involutive if any two vector fields belonging in the distribution implies that the re-bracket belongs in the distribution.

Okay. Okay. So we are asking for more things now. Okay. We already talked about distribution and non-singularity of the distribution. Now we are saying what it means for a vector field to belong to a distribution. That is pretty straightforward.

Right. That if you plug in any point, then f of p belongs to delta of p. Right. And then you say that distribution is involutive if there are two vector fields belonging to the distribution means that re-bracket also has to belong to the distribution. Notice the re-bracket is not some linear combination or anything.

Right. So the re-bracket is a sort of a bracket operation. Okay. But we know that the re-bracket also gives a new vector field. So what are we saying? If two vector fields belong to a distribution, this modified new vector field also belongs in the distribution.

Okay. So this is actually asking for a lot. This is called involutivity. Yeah. Entire results on controllability are based on involutive distributions. Yeah. Remember I told you, right, once again all of that was very vague because we did not prove anything or use it anywhere. But I told you, right, that you do not just move along linear combinations of these.

You also move along re-brackets of these. You can also move along re-brackets of these. Okay. So that is the cool thing. Yeah. So that is why we are interested in involutive distributions in general. Alright. So we have a very nice result which really helps us to identify which distribution is involutive in a rather easy way.

Okay. What is the result? It says a non-singular distribution generated by smooth vector fields, this, is involutive if and only if the lie bracket between fi and fj is in the distribution. Now you might say it seems like I have asked for everything.

But actually I have not. Okay. Actually I have not. Okay. If I just go by this definition, right, if any two vectors belong to the distribution, any two vector fields belong to the distribution, the lie bracket must belong to the distribution. That is what is involutivity. So if I have f1 to fk, there are k such vectors.

Can you imagine how many combinations I have to check? Many many. Yeah. Yeah. Because it is not that simple. It is not enough to just check this much. If I was looking, going just by the definition, so f1, f2 has to be in the distribution.

Right. f1, f1, f2 has to be in the distribution. f3, f1, f2 has to be in the distribution. You see how it is going. This is only the second successive lie bracket. I have to go further further further. Right.

Many many such successive lie brackets because every time I do a lie bracket I get another vector field. Right. So you can imagine you are going just by the definition, checking all of these impossible would make my life hell.

Alright. So that is what this lemma does. It gives you a very nice simple characterization. It says that I don't have to check all of these lie brackets and the successive lie brackets and so on and so forth. I only have to check this for every possible combination of these k vector fields. Thank you.