Nonlinear Control Design

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Week 9 : Lecture 54 : Feedback Linearization: Part 6

So, let us look at the comparison with the linear system case alright. So, remember we had a single input system that we are working with and a single output system, single input and a trans-single input. So, let us look at a linear single input single output system. So, can be represented by a transfer function right and it has a standard canonical state space realization also. All of you are aware of this I hope yeah from this transfer function I can get to this state space realization right. So, basically these two are equivalent.

So, this is the A matrix, this is the B matrix, this is the C matrix. B is the control matrix, A is the was the drift and C is the output ok. From this transfer function this is what will be your minimal realization standard ok. Why did I move to the A B C's because of course I want to work in state space right not in frequency domain.

So, with this setup you can see that the entire system is linear. So, there is no partial linearization and all going on here we are just trying to see what happens. So, if you look at Z1, Z1 is y equal to this entire thing right. So, I expanded it right. It is x1, x2 all the way to xn minus r plus 1 yeah.

So, I am just how do you get the output? Output is just this multiplied by x the states right. So, that is how I get the output ok. Z2 is obviously y1 dot just how it is been defined right. So, or sorry Z1 dot if you want to write sorry Z2 is Z1 dot because it is y dot and you can expand it, but it is not relevant. Basically you have all the way going from Z1 to Z2 wait a second yeah.

So, Z1 dot is Z2, Z2 dot is Z3, Zr minus 1 dot is Zr. So, that is the first whatever the first r minus 1 columns covered ok that is this guy I see. So, this is only using n minus r and this is of course n dimensional ok yeah that is fine. It is just how the transfer functions. Now it is very deliberately done the numerator is deliberately has power only to n minus r just so that you have zeros in the end.

It is not it is not making use of all the states ok, but this is obvious just because of this dynamics if you write this as x dot is Ax plus Bu if I write it like this or actually in this case the way it is been written is not x dot, but it is written as z dot is Az plus Bu we are using z not x. So, if you just write this out z dot is Az plus Bu you will get this right Z1 dot is Z2 you will get this part right it is pretty obvious right. The only thing left is the last one ok actually not the last one not not even the last one I apologize not even the last one. I am just trying to see what is happening. So, Z1 dot is Z2 and Zr minus 1 dot is Zr that is obvious and what are we doing we are writing Zr as this guy I see what is the need to do this why am I doing this more right sorry I am going to move forward yeah this is I am just going to move forward and I will come back to it if we need it yeah.

This much is obvious from the dynamics z dot is Az plus Bu right. In fact the u appears only in the Zn dot we are only going from 1 to r minus 1 here ok. Then we want to look at Zr dot ok that is that is where all this is happening ok what is Zr dot now? What is Zr dot? From this expression itself you will see that Zr dot is simply equal to Zr plus 1 because I have not reached n ok I hope that is evident ok yeah I have not reached n the r and the n are different ok so and r is obviously less than n so Zr is somewhere here. So, so just by this dynamics itself yeah Zr minus 1 dot is Zr similarly Zr dot is Zr plus 1 ok. Now I am just wondering how we get to this form ok with the control that is all how does Zr dot contain the control I guess I have no choice but to actually look at this disappointing alright we have to look at it this way there is no choice ok.

If you look at Z1 it is y which is exactly this way which is expanded here ok Z2 is Z1 dot I am actually taking the derivatives here so x1 dot gives me x2 x2 dot gives me x3 xn minus r dot gives me xn minus r plus 1 and xn minus r plus 1 gives me xn minus r plus 2 ok I have no choice but to do this yeah and you keep doing this onwards and onwards if Zr is actually equal to Zr minus 1 dot right which gives me what you can keep continuing the same logic x2 will go if you notice in 2 there is 2 here right. So I will get the r index here similarly the r plus 1 index here yeah so you just add basically r to it so you will get n minus r plus 1 will give change to n minus r sorry n minus 1 and this will become n ok is that clear basically I am not doing this is nothing no magic here yeah it is very yeah let us not use that forget that I mean it is but it is useless I mean because Zr plus 1 is coming from Zr so see how Zr plus 1 is not actually this is not true this is not true I should erase this actually this is not what we do we do not define Zr plus 1 as Zr dot yeah that is not correct ok. So because we define new coordinates the phi coordinates yeah these are only the linear coordinates as of now we are only looking at the linear coordinates so how we are getting to the dynamics of the linear coordinates is simply by taking consecutive derivatives of y just like we have been doing ok and if you keep taking the consecutive derivatives of y this is what you will get yeah you go to b0 x2 b1 x3 and so on and then if you keep moving forward in all the way to Zr which is Zr minus 1 dot you will just match the indices here 2 becomes 2 3 becomes r plus 1 and so on and so forth yeah basically all you are doing is you know you are basically just moving from the 2th index to the rth index that's it you can see the pattern here it is nothing complicated and then the only important thing to remember here is this brings in xn that's it see this last term did not mean anything earlier but now you have an xn why because xn dot contains the control this is what is important ok. So when I take Zr dot which is the derivative of this guy these things won't do anything this will give me r plus 1 r plus 2 n but xn dot will bring in the control as it should why because it is a relative degree r system yeah ok or if it was not evident to you earlier just by the fact that you go only till n minus r here in the numerator makes it a relative degree r system ok for the linear case. So this is actually relative degree r system even if it was not evident to you earlier by taking these derivatives it becomes evident to you right because the control appears only in the rth derivative so obviously it has to be a relative degree r system yeah so that's it I have a relative degree r system this is my linear dynamics Z1 dot is Z2 Z2 dot is Z3 all the way and then Zr dot contains the control ok that's what is written here that's it some r and s matrices we need not concern ourselves with it but there is the control here.

Now how do we choose the rest of the coordinates? Well the way it's been chosen here in these nodes is just taken to be the first n minus r states right because I already had Z1 to Zr so now I need n minus r more states so I am just taking them as the first n minus r states ok it turns out to work fine so here it says you have to find the rank of the Jacobian yeah I can even put this as an exercise yeah it's pretty easy to check yeah because the first because the rest of the states are these Z1 to Zr and this is the X1 to Xn minus r yeah if you verify the Jacobian will turn out to be full rank alright. Now so now what so this is what is the additional dynamics right this is what will give me the zero dynamics alright and this is what is the etas right so the Z1 to Zr form the size so the etas are these guys ok so what is the derivative of the etas? X1 dot is X2, X2 dot is X3, Xn minus r dot is Xn minus r plus 1 ok alright. So if you want to write them in terms of the eta and xi X1 dot is X2 so this is eta 2, X2 dot is X3 eta 3 this is how I have actually constructed the etas and this guy yeah then you go all the way to eta n minus r and this guy Xn minus r plus 1 ok what is Xn minus r plus 1? It is not any eta I hope that's evident right because eta goes only to Xn minus r right it's not eta so how do I write it? I just go back to this equation right look at this yeah you can see Xn minus r plus 1 appearing here alright. So I am only trying to write everything in terms of the new variables that's all I am not doing any magic here I am just trying to write everything in terms of the new variables ok. So I know that X1 to Xn minus r are my eta 1 to eta n minus r and then I have Z1 to Zr so all I want to do is write my zero dynamics in terms of these eta and Z so eta and whatever Z1 to Zr eta and Z1 to Zr alright.

Now my problem is when I took the eta dot the last term came out to be Xn minus r plus 1 which is not eta and it is not directly any Z1 to Zr either ok. So but it's pretty simple I just I can see that my output Z1 contains the Xn minus r plus 1 right. So I can write this as Z1 minus b0 eta 1 b1 eta 2 bn minus r minus 1 eta n minus r right. So I can directly write this in terms of Z1 and etas right just by using the output equation yeah. So that's what I have it is xi 1 which is Z1 minus this entire alright ok.

That's what I have written yeah you have a p xi and a q eta and q is basically this coming from here p xi plus q eta and the q comes from this guy right and so if xi is 0 your 0 dynamics is just this much which is this yeah ok clear. And so what is this? If you look at this equation this is first of all what r dimensional sorry n minus r sorry keep reminding me ok. So this is n minus r dimensional system so this is also obviously n minus r yeah but if you look at the structure of this what are the poles of this system? What are the poles? They are basically sorry not the poles but the coefficients of the transfer function are this right of this right alright ok. But so if you want to find the eigenvalues what you have to do? You have to basically find the zeros of the characteristic equation which is being going to be defined by this guy just this guy right. So basically the eigenvalues of this system are the zeros of the original transfer function right.

What was my original transfer function? Was just this right right. So zeros of this guy is actually equal to eigenvalues of my 0 dynamics ok. Zeros of the original transfer function are the eigenvalues of the 0 dynamics right. That's what you have q is this eigenvalues of this is exactly defined by zeros of that transfer function because if you write the characteristic equation of this matrix q you will get exactly these as coefficients right alright. So what do you have? The 0 dynamics the name has a very clear connotation ok.

It is coming from the linear system context yeah that it is the zeros of this transfer function is what is giving you the 0 dynamics ok. Pretty interesting actually right. So depending on how this guy looks if you are looking at a linear system depending on how this guy looks to you, you know that what is the first of all you will know what is the relative degree of the system right. In this case what was the relative degree of the system? R which was n minus the highest power of this yeah. Relative degree is just n minus the highest power.

Why? Because I will keep taking successive derivatives right that's what I did and I will get to the nth power after r minus 1 steps and then if I take one more I get the control right. So relative degree is defined immediately by looking at this and the zeros of this guy yeah or the solution of this gives you the eigenvalues of the 0 dynamics yeah. So if this is a stable system this represents a stable system ok then so if this equal to 0 gives me all real negative real parts negative for the corresponding s then I am good right basically gives me that my 0 dynamics is stable right and that in frequency domain context what is this called when the zeros are in the left half plane? Minimum phase exactly minimum phase ok so and that's the nice property that we are extending to non-linear systems right. So this is a minimum phase system. So 0 dynamics having stable a stable 0 dynamics is equivalent to having minimum phase in the linear system ok.

So in this case if you have a minimum phase system your Q will have negative real eigenvalues or whatever will have eigenvalues with negative real parts ok which is a exponentially stable system and that's what you want remember yeah I hope you remember because we can't do anything with the non-linear piece. So if you have negative real eigenvalues with negative real parts for that amazing right. So it's going to 0 exponentially fast right. In this linear system context also I cannot have played with the eta dynamics done anything to the eta dynamics right but because it turned out that this has good structure yeah I can now only work with the z dynamics yeah alright ok great. So we don't have to talk about this right.

So that's the idea ok in order to get asymptotic stability for the entire non-linear system which has been partially linearized alright you need the non-linear part or the 0 dynamics to have nice features which is stability 0 dynamics should be stable alright that's really the idea and that's what we are sort of trying to state here alright. And what does it say? It says that if you have if you without loss of generality if you assume that this is an equilibrium of the system psi and eta equal to 0 0 that is in the z states right. And it has relative degree r ok then if the 0 dynamics is locally asymptotically stable ok. Suppose that the equilibrium eta equal to 0 of the 0 dynamics is locally asymptotically stable or you also say use these words the system is minimum phase yeah and the feedback linearized subsystem is also asymptotically stable right. So what are we saying? The linear part is asymptotically stable or stabilized by the control that you have ok and on top of that the non-linear part that is the 0 dynamics of the non-linear part is also asymptotically stable then the combination is asymptotically stable ok.

This is basically the cascade idea because your system looks like this you have a linear part yeah just by virtue of your control you have a linear part yeah and you assume that this is exponentially stable in fact this is exponentially stable right and then you have a non-linear part alright you have a non-linear part which I know is locally asymptotically stable when the linear part is 0 this is the 0 dynamics right 0 dynamics is this guy and I am saying that this is locally asymptotically stable. So the combination is in fact asymptotically stable that is the claim ok. Any questions? Yeah. How the Q? How do I write the Q? Ok.

That is fine that is fine. So we are writing this equation by the way not this. We are actually writing this equation. I hope you got that eta is this x1 x2 all the way to xn minus r. Now all I am trying to do is because this is the eta dynamics right. So I am just trying to write eta dot from here.

So eta dot is x1 dot is x2 so eta 1 dot is eta 2 and so on. So x1 dot is x2 x2 dot is x3 that is what I have written x1 dot is x2 x2 dot is x3 xn minus r dot is xn minus r plus 1 alright then x2 is actually eta 2 x3 is eta 3 and so on and so forth then I will get eta n minus eta n minus r. We will not if you see here. You have defined it. Yeah and that r plus 1 to zn is defined as this guy.

See eta eta and psi are just the split of the z vector that is all. I mean I know I am introducing not of new notation but the way we have been working is we move from the x variables right from the x variables we move to the z variables and all I am doing is splitting this z into psi and eta because this is the linear yeah I want a linear piece and then this is the non-linear piece that's all that's what I am repeating here also yeah. So psi is basically the first few vectors of z which was the linear part already done and then this is the non-linear part which I am defining these coordinates I have to define these are the phi's phi i's these I have to define these I am defining at these vectors these variables from the original state yeah these I have to write. So now eta 2 is sorry x2 is eta 2 x3 is eta 3 yeah then I go all the way to eta n minus r but then xn minus r plus 1 is not in eta yeah to do so to write xn minus r plus 1 I go back to my output equation right here I have from this guy I can write xn minus r plus 1 as z1 minus b0 x1 minus b1 x2 all the way to minus bn minus 1 xn minus r and I know that this is eta 1 this is eta 2 this is eta n minus r right and this is z1. So if you look at this z1 is basically coming from the linear part from the xi so I have xi term and then zr plus 1 to zn is coming in the eta term this I have just written as this guy right q eta that is the q because I don't care about this because I am going to make this 0 so I just write the q from here actually here yeah I skip the xi part that is all that is how I get the q this is just 0 if you look at this 0 1 0 0 1 all the way to you know there will be a 0 0 there will be a 1 in the end and then I will get this guy yeah alright. Thank you.