Nonlinear Control Design

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Week 9 : Lecture 53 : Feedback Linearization: Part 5

Welcome to our class on Non-linear control. So, let us go back to where we were. We had started talking about feedback linearization. And we were actually doing a couple of, of course I mean we introduced some notation, which probably was a little bit complicated. But then we also did an example. So, hopefully there is a little bit of clarity and you also hope that some of you did go back and actually try to look through this material.

I posted it already on Moodle. So, the lectures are already on there. So, this is what I have been using and I plan to use subsequently also. So, the only two real notations and then of course extended versions of this.

There is the Lie derivative, which is basically the directional derivative of a scalar valued function along a vector field. And then there was the Lie bracket, which basically takes two vector fields and gives you a sort of a, sort of a skew symmetric operation. It is like a DGF minus DFG and this kind of an operation. And then there was of course the add notation, which is basically just a different way of writing iterated Lie brackets. We had used this to of course look at systems of this form with output which looks like this.

And we want to sort of come up with this diffeomorphism, this transformation. If so, if you remember the feedback linearization has two pieces. There is a state linearization or a state transformation and there is a feedback which is then finally used to linearize the system or make the system look linear. I would not say linearize. So, the idea is you just take successive derivatives of this output and you try to see in which derivative the input shows up and that is called the relative degree.

And that is basically qualified with this sort of expression because as you take derivatives of y, the time derivatives of y, you will start seeing the dynamics again and again and again. And that is what happens. You have y is h and then you have the first derivative is Lfh. If you assume that the Lgh is 0. Similarly, the second derivative becomes Lf squared h if you assume that Lfh is 0 and so on and so forth.

And this does happen because you assume relative degree r. So, this happens until k equal to r minus 2. After that you get something non-zero. So notice that we are still working with a scalar valued control. This is a single input system.

The idea is again work for the multi input case but this thing is just become a little bit more complicated. So, easier to work here with the single input case. Alright, so that is the idea. We essentially try to compute the relative degree by seeing how many derivatives of the output you need to get to the input. And we then say, sort of say that this is how, this is the part of the dynamics that you can actually linearize via feedback.

So this r, the dynamics of size r is what you can linearize. So it is like a partial feedback linearization and the rest of it is some additional dynamics. So in order to get to this sort of nice linear sort of structure, we need a few inequalities. That is in order to verify that these things, this y equal to h, its derivative, its second derivative, they do form new coordinates, we need some equalities. The first one basically said that if you have these qualities to be 0, that is Lgh, Lglf, Lglf square all the way to Lglfk to be 0, then it is equivalent to saying that Lg, Laddfg, Laddkg all the way is 0.

Okay, these two are equivalent. Okay, so and the key, the very very key identity that we use to prove all of this is basically this guy. Okay, this is what you need. Alright, this is what you need. So I can even highlight this.

Yeah, so this is really what you will require in order to complete all of this proof. We actually looked at the proof. Okay, so once you have that this sort of an iterated relationship, yeah, there is an equality that is between Lg, Lfk to Laddfkg. Yeah, there is a similarity here, there is an equivalence here. You can actually start talking about independence of these vectors.

Okay, what are these vectors? This is just the Jacobian corresponding to this new coordinates. That is the new coordinates are h, Lfh all the way to this Lfr minus 1h. Okay, and so if you take the d which is just the partial, you get a Jacobian structure. Alright, so because dh is a row vector, dLfh is a row vector, so you essentially get, you know, a Jacobian. And we used, we want to prove that, we want to prove that this is linearly independent.

Okay, these r row vectors are in fact linearly independent. And how did we do that? We did that by looking at the multiplication of this guy with some other vector. Okay, which was very smartly chosen. Okay, why was it smartly chosen? Because once I do the multiplication here, we actually did it very carefully. Right, once I do the multiplication here, you can see that every row basically just contains, you know, things like this, this guy.

Right, and then this guy and so on and so forth. So basically you start seeing a lower triangular structure. Alright, why? This is because of the first equality that we proved. Okay, this is just coming from the first equality. Right, in this case you have this last term to be non-zero, in the second row you have last two terms to be non-zero and so on and so forth.

So you have a lower triangular form. Alright, and it is well known that the lower triangular form, whatever is the number of non-zero rows is the rank. Okay, so the product of this matrix, these two matrices is an R by R matrix and we have proven that the R by R matrix has rank R. Okay, which means individually each of these guys also have to have rank R. Right, and therefore these are linearly independent and these are also linearly independent.

Okay, so you simultaneously prove that all of these are in fact linearly independent. Okay, alright. So this is where we were. Yeah, of course with the linear independence you, once you have this linear independence you can construct this change of coordinates. Right, and since this gives me only R number of coordinates, I actually add a few more.

What is how many more? N minus R more, because I started with an N dimensional, you know, state space. So I have to go to another N dimensional state space. So I have this R and then I have this N minus R where these phi are simply chosen to make sure that this whole map phi, capital phi is a diffeomorphism. Okay, which essentially means that the Jacobian is invertible. Okay, has to have an invertible Jacobian.

Right, so that is essentially what it, this says. This guy has full rank. Okay, yeah, full rank means invertible in this case. Yeah, because you can see that phi is, it's N dimensional.

X is also N dimensional. So when I take the Jacobian it's an N by N matrix. Right, so full rank means it's an invertible matrix. So del phi del x or d phi, whatever you want to write it, is supposed to be an invertible matrix. Now in order to help with that we already know that this, the Jacobian of this is invertible or full rank, sorry, not invertible, in this case maximal rank. Right, why? Because of the lemma 0.

2. Lemma 0.2 basically just said that, that this is maximal rank or this is rank R. If these rows are rank R then obviously the Jacobian of this is exactly this guy. Right, and that's also rank R. Okay, so this helps us obviously.

So lemma 0.2 is what helps us in claiming that these are in fact independent coordinates. This is what it means for the coordinates to be independent. I mean we are not used to, I mean we are not used to this because we just say that x1, x2, x3 are our coordinates and we think they are independent. Because they are in orthogonal frames.

Right, so we never think about it. But now if I transform this x1, x2, x3 non-linear to some other function, other three functions. Okay, so I say x1 square plus x2 square minus x3 square x1 square plus x2 square plus x3 square x1 square minus x2 square plus x3 square. I just gave you some non-linear transformation. Now what is the guarantee that these new coordinates are in fact coordinates, that is they are linearly independent. Okay, the only way to claim that is by checking the Jacobian.

Okay, you have to take the d phi. Okay, and then you have to see that it is full rank. Right, and if it is you are good. And that's really what we are trying to do. So because of these are independent coordinates, because it gives me maximal rank, not full rank in this case, but maximal rank. So all I have to do is to make sure that the d phi of these, that is partials of these are also linearly independent.

Okay, that's it. Okay, and we actually looked at an example, like this DC motor example. Right, what did we do? We actually verified both the lemmas. Right, first is this equality lemma, which is basically saying that you know this LGLF and LGLF all the way to LGLF r minus 2 is 0, which means that you want to prove that L add f r minus 1 is 0. Okay, sorry, L add f r minus 2 is 0.

Okay, and that's what we did. Basically, it's easy to see that L add f k is 0 in this case, because what does r turn out to be? What was the relative degree of the system? The relative degree was just 2. It was very simple. We just we did all the computations. What did we do? We computed LGLF, first we computed LGH, it was 0. Then we computed LFH, which was theta x1 x2.

Then we computed LGLFH, which was theta x2. So this itself came out to be non-zero. Right, so this is just the first power. So basically you have r equal to 2. Because this 1 is equal to r minus 1.

So r is basically equal to 2. Okay, so we got relative degree 2. Right, and so all we had to prove was that L add f 0 GH is equal to 0 and that's what we proved. It's very easy, already done.

Yeah, so lemma 0.1 was easy to prove. Then we wanted to look at the rank of you know the lemma 0.2, which is basically saying that the H and LFH, which are our new coordinates. So in this case, H is basically your x3 and LFH is theta x1 x2. These are the two new coordinates. Right, and we wanted to see if there Jacobian has maximal rank.

So we computed that. Right, we actually computed the Jacobian, it is DH is 0 0 1 because H is x3. Right, and then LFH is theta x1 x2. So DLFH is basically this guy. And for this to be maximal rank, we just need for x1 x2 to be non-zero. Okay, then the way we chose our phi that is the third coordinate was just to make sure that I get a full rank Jacobian.

Okay, and what did I recommend? I thought, I just thought I will make the third row orthogonal to the second row because this first and the second row are already orthogonal. Notice, because the dot product of first and second is 0 already. Right, the first and second are already orthogonal. So I just made sure that the second and third are orthogonal.

So to do that I chose this sort of a y3. Right, so everything looks non-linear. And of course now I know that if x1 x2 is not equal to 0 0, then this is a full rank matrix. Right, so I am good to go. This is a valid new set of coordinates.

Valid new set of coordinates. Okay, the other choice which I had sort of used earlier was taking the third coordinate as x2 minus b. Right, in that case I would have got 0 1 0. But actually I have written it here. Yeah, corresponding to this I would get the third row as 0 1 0.

Yeah, this is fine too. The only problem is in this case we saw that you need x2 be non-zero for this to be full rank. Yeah, if x2 is 0 then you see that the second and third rows are exactly the same. Alright, and that is a problem.

Okay, so if x2 is non-zero all of this works. No problem. This new coordinate also works. There is value in looking at this. We will look at, we will see this immediately, subsequently.

Okay, alright. So now, great. We now are in a position to sort of talk about the transformed system. What does it look like? So if you look at this, this sort of new representation or new variables in which we are writing the dynamics. Okay, and you start computing the derivatives. Right, you start computing these derivatives. This is basically, well I mean this is written in terms of this guy.

Right, that is z is basically looked upon as all these coordinates. These new coordinates are just written as z.

I used y. That is not a big deal. Okay, alright. So this is just writing in terms of the capital phi. This is not very useful to us. Just look at this part. Okay, z1 is basically the actual output of the system.

So z1 dot is actually LfH has to be because Lgh is 0. That is how we have been doing. And that is equal to z2 itself because z2 is LfH. Now if you take the derivative again of z2, it is d dt LfH but that is going to be Lf squared H because again Lg LfH is 0. Right, this is again how we have got the relative degree.

This is by the relative degree assumption. Right, so therefore you have z2 dot is Lf square H which is z3. Okay, and you can keep on going like this. You essentially form an integrator, a chain of integrators. Okay, z1 dot is z2, z2 dot is z3 and so on and so forth until you get to zr dot.

Right, which is now the last coordinate, this guy. Now when I take the derivative of this, I will get what? Lf Rh plus Lg Lf R minus 1H. Right, just by taking the standard derivative. Right, and plugging in the dynamics. Okay, now you know that this is not 0 because of our again relative degree R assumption.

Right, so I am going to write this as some Bz plus Azu where Az is not 0. Okay, so that is what the linear part looks like. Right, it is just a bunch of integrators and then the final state has some nonlinearity in its derivatives. Okay, so notice I started with a nonlinear system which was nonlinear everywhere.

Right, in all the states probably. Yeah, just like in the DC motor case. But now I have because of my state transformation, I have reached a stage where I actually have linear integrators everywhere and only in the last state there is a nonlinearity but I will just define this as my new input V. Right, and that is possible because U is actually V minus B divided by A and A is nonzero. Right, therefore I can do this assignment. From V I can compute U, no problem. There is no singularity issue or anything because A is nonzero by my relative degree assumption.

Okay, therefore I just had a chain of integrators here. So this is the partial linearization that we have achieved. Okay, you cannot do anything more. Whatever is the relative degree of your output, that is the best you can do. If you can find an output for which your relative degree is n, then this entire thing will look like a bunch of integrators. Okay, I hope that is clear that if relative degree is equal to n, then this is just all going to be looking like a bunch of integrators.

Okay, so you have effectively completely linearized the system. Okay, alright, great. Now for the rest of the dynamics, if we assume that the phi is, okay, until now the way we were choosing phi was just to ensure that you got a diffeomorphism, that is you got Jacobian full rank which is what we did here. Right, we just gave a row which sort of made sure that, you know, this is a full rank Jacobian and from that I went back and constructed the additional phi states. Right, but here we are saying, suppose I have another assumption on phi, that in the phi dynamics the control does not show up.

Okay, so that is the assumption. So that is what is the normal form. If control does not appear in phi i dot. Okay, the control does not appear in phi i dot equations, then that is how you choose the phi, which means, what does it mean? It means that Lg phi i x is equal to 0, that is the way control will not appear. Right, this is the control vector field. Right, if I take phi i dot, I get Lf phi i plus Lg phi i times control.

Right, so if Lg phi i is 0, then no control appears in the phi equations. Right, and with such choice of phi, if you wrote this dynamics, you have what is called a normal form. Okay, and the d dt phi i is Lf phi i and we just call it Qiz, because Qi is just a new notation, because now we are writing in terms of the new variables, z, that is all. Alright, so what do I have then in the normal form? In the normal form, I again have this bunch of integrators, that does not change. Here, control does not appear anymore.

Okay, control does not appear here, because I choose the phi's in a smart way. Right, so that phi dot does not have the control. Alright, does that make sense? Okay, now if you go back now to our example. Okay, you look at this choice of y1, y2 and then y1, y2 is what they are, we can't really play with them.

Yeah, we chose this guy. What is y3 dot? This guy. Yeah, but control appears in that equation. Right, so this choice of transformations does not give me the normal form. Okay, does not give me the normal form. It just gives me a linearized form, but it's not the normal form.

Usually in feedback linearization, control theories prefer to work with the normal form. Why? Because it's just nice, right. You have a non-linear system, right, whatever nonlinearities there are, but there is no control in that, but then you have a linear system sort of driving this nonlinear system. Alright, and this driving system is linear. Okay, so you can do a lot of things with the driving system.

Right, remember again with the cascade, go back to the cascade idea. Right, there was a driving system or a driven system. So here this will become the driving system and the output of this will go to this way because we will look at it, how that happens. But the point is there is nonlinearities here with no control and then there is this linearity here where you can control these states very well. Basically you can make sure these states do whatever you want.

Yeah, because it's linear and has a very nice structure. Okay, so this is the normal form. What I am trying to say is what we chose as y3 does not get us to the normal form. This on the other hand gives us the normal form. Why? Because if you look at z3 dot, okay, what is z3 dot? z3 dot is just x2 dot.

Right, what was x2 dot? No control. No control in x2 dot. Okay, so but remember this was a restrictive choice, right, because this was a valid choice only when x2 is non-zero. Okay, this is a valid transformation only if x2 is non-zero. Okay, so it's a slightly more restrictive choice. Here we had the freedom of having either x1 or x2 non-zero.

Anything being non-zero was good enough for here, but here no. We definitely need x2 to be non-zero. But although this is a restrictive choice, it still gives us the normal form. Okay, which is why this is also an important sort of transformation. I think that's what I wrote here.

Yeah, this transformation for the DC motor system is what gives you the normal form. Okay, it gives you z1 dot is z2, z2 dot is this guy and z3 dot is this.

No control here. Notice, the control doesn't appear here. Okay, alright. So that's how you do it. You get to the normal form. In addition to trying to get to a diffeomorphism, you make sure that the, in the derivative of the extra states or the phi states, the control doesn't appear. Alright, so that is how you get to the normal form. Alright, so now you have essentially what is called a partially linearized system.

Yeah, because like I said, with this choice of V, you have the z1 to zr states being a linear system. Right, just bunch of integrators actually, not any linear system, but a very specific linear system. And then you have a bunch of non-linear systems. Alright, we'll see what can be done in these cases. Yeah, how to control in these cases.

But before we do that, we want to define the notation of a zero dynamics. Alright, so we are denoting by xi these partially linearized states, which is z1 to zr, and by eta the rest of the states. Okay, just for notation sake. So the xi dot system is again in this form, integrator form.

Right, so in this integrator form. Yeah, so and the eta system is some non-linear form. Right, we don't know what it is, but it is some non-linearity. The only thing that we know is because of its normal form, the control doesn't appear. Right, so it is some function of xi and eta. So what we have done is we have split the z states into xi and eta states.

Okay, so I have split the z states into the xi states and the eta states. That's all. Yeah, the xi states correspond to the linear part and the eta correspond to the non-linear part. Yeah, now suppose that the output is identically zero, then all its derivatives are also identically zero. Right, which means xi is identically zero. Okay, what this does to the xi dynamics or the, whatever, I mean the way, so if you look at the zr dynamics, so zr dot is also zero.

Okay, and that essentially is equal to this guy. I mean you are just plugging in xi equal to zero in this right hand side. Yeah. No, because this is depending on all the states.

See, if you go back here, let's go back here. All this entire state, right, entire z, I can't control that. That is coming from all this Lfh and Lgh. Yeah, that you can't control. It is not just depending on these states.

Yeah, it can depend on both. So all I am doing is I am splitting the z into these two pieces. All right. Right, so anyway this is anyway too much detail to get to the basic point. The basic point is that what you call the zero dynamics is when xi is zero here. Okay, remember this is the nonlinear part and if you put xi equal to zero here, what you get is the zero dynamics. Okay, why this is of interest, why this is of value is because this xi system is the linear system.

Right, the assumption is that I can do anything with it. So I can even drive it to zero as fast as I want. So if you remember, even in the cascade case, right, what did we say? That we had a stable system which is being, which has an additive term which is the, which is basically coming out of a passive system, which is the output of a passive system. So we were putting a nonlinear stable system in cascade with a, right, you had this guy. So here what did you have? You had a stable system here, right, and you had in addition to it this basically this y guy that was coming out of a passive system. Right, it's very similar to that. Here if I make this zero, right, that was the whole idea, right, if y is zero, right, then this system is just z dependent and this is a stable system.

Right, but we also know that I can do nice things with y because of passivity in this case. In this case it was passivity and that is what is driving this system. So similarly you have this idea that this xi states not passive in this case but they are basically a linear system, coming from a linear system. So the assumption is I can do whatever I want with it. Therefore it is important to actually study the zero dynamics. That is what happens or how does this system, the nonlinear system behave when the linear part goes away or decays or dies down to zero, what happens, okay, and that's important, all right. Thank you.