Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 9 : Lecture 51 : Feedback Linearization: Part 3

We just rewrote this guy here. We just rewrote this basically the expression in terms of add f g's and so on. Basically the equivalence of the add f g's with lglf's. This equivalence is what we have rewritten here. Why we have rewritten here? Because we are going to use this in the matrix multiplication to show that some terms are zero and some are not. So what is this matrix we are multiplying? We took that all the row vectors we had, we stacked them.

So I got a R by N matrix. R rows because whatever R terms and N columns because of course partial of H with respect to X will have N columns. And I multiply it with these guys. You start seeing all the add notation here.

And this is a N by R matrix because each is a vector field. Each term is a vector field. It's an N by 1 vector and there are R such terms. It's an N by R. So the product as you can see is a R by R matrix.

It's a square matrix. Deliberately done. Very deliberately constructed. Now if you look at some of the terms in this product. You already know this is an R by N matrix.

One of them is an R by N. Other is an N by R. So the product is an R by R matrix. That's obvious. Now we want to look at the rank of this product.

So obviously we have to multiply things out and see what happens. So this is deliberately chosen in a very smart way so that it turns out to be a nice structure. So what will be the first row? Row 1 will be DH. I am going to remove, forget the X0 argument. So I have to write this.

DH times G or I can even write this as inner product of DHG. DH add FG. DH add R minus 1G. This is row 1. Everybody agrees? I have just done the product.

You can see these are consistent because this was a, each row is an R by 1 vector and each, sorry, 1 by N vector. This is 1 by N multiplied by N by 1. So I get a scalar. Inner product gives you a scalar always. So that's it.

What do I know from here? This thing that I concluded, this complicated thing, this contains all this, no? This is all this mess. Yeah. If you look at the first term and the second term, DHG, we just evaluated this. DHG corresponds to, in fact I am going to write this out also if I can.

Paste. Yeah. I am going to use this. What is this? It's saying that whatever, this mess is actually equal to this and I also have obviously from the previous lemma that and the fact that we are relative degree R system that this is non-zero only when this is R minus 1 and it's 0 for K plus L is less than R minus 1. So if I look at the first term, can you tell me what is K and L? Yeah. Can you tell me K and L corresponding to this? I will just use IJ here and K and L here. They are just replaceable, right? Whatever is K here is I here, whatever is L here is J here.

Same thing. Okay. So in this case what is it? J is 0, right? You just did this, right? J is 0, I is 0, right? So K is 0, I is 0, right? So their sum is 0, right? Which is less than R minus 1. It is less than R minus 1. So basically what happens? Their inner product is 0.

Great. What about the second one now? Again, in this case J is what? What is J from here? J is 0. Excellent. What is I? 1. What is their sum? 1. Again less than R minus 1.

So inner product is 0. Keep going on. You will get 0s everywhere. Let's look at the last term directly. Let's look at the last term.

What is the J here? 0, right? Because I didn't change the first term at all. J is 0. J remains fixed. I is R minus. What is the sum? R minus 1.

Non-zero. Non-zero. So I am going to say this is just whatever. I am just going to write it as L whatever. Add L add F R minus 1 G H. Rho 2.

Let's look at rho 2. What happens to rho 2? It gets replaced by L F H. That's it. Everything else is the same. So what is it? Inner product D L F H G inner product D L F H add F G all the way inner product D L F H add F R minus 1 G.

Ok great. First term. What is J? 1. Remember in every row J remains fixed. Excellent. So I don't have to compute J again.

J is fixed now. J is 1 for this row. Ok. I is going to change of course. I is 0. 1 plus 0, 1 less than R minus 1.

So inner product has to be 0 by this. Again second term 1 and 1, 2. Ok. Let's look at the last term. In the last term what is again J is 1.

What is this guy? R minus 1. Sum is R. So this is not equal to 0. Right. So there is some actually I am going to make my life simple and I am going to call it not alpha right. Yeah I am going to call it alpha 2 1 which is non-zero.

What about the term before the last one? What is the term before the last one? dLfh add f R minus 2h. Non-zero. That is because the sum is R minus 1. Right.

Because J is 1, I is R minus 2. Right. So again I have R minus 1 which is non-zero. So I have something which is alpha 2 2.

Yeah. I am just using some notation. I am going to call this alpha 1 1. Yeah. So that I have to write the whole thing. Now you can go on and on.

You can see what is going to happen. For the third row J will be always 2. Excellent. And so the first term will combine to give 2 plus 0 2, 2 plus 1 3 and so on and so forth. So the last three terms will be non-zero.

Very nice. We are getting the pattern. So row 3 will be alpha 3 1, alpha 3 2, alpha 3 3. Yeah. And zeros everywhere of course. Yeah. What happened? I get what is being claimed here.

I get a lower triangular matrix. Like this. I get a lower triangular matrix. And what is the cool thing about a lower triangular matrix? What is the rank of this lower triangular matrix? Well it is an R by R matrix.

It is R. Lower triangular matrix rank R. Right. Because very easy. Right. Find null space. Because this is something multiplied by nth term is 0 means nth term is 0.

Last two multiplied by last two terms is 0. Whatever. It is in echelon form without having to reduce. It is already in echelon form. So rank is obviously the number of non-zero rows.

Ok. It is rank R. That is it. So that is what we wanted to show. That it is rank R. So here I will say lower triangular.

Echelon form. Yeah. We have rank R. And then this is all this explanation. Yeah. Which we don't care about. I already explained it to you.

Whatever is written here is irrelevant. We have already discussed it. Yeah. What have we just shown? That the product of these two matrices is rank R. Which means the product of the individual matrices has to be at least rank R.

Ok. Now notice what were the individual matrices. This guy and that guy. What are the dimensions of these matrix? R by N and N by R. Right. So each of them can anyway be maximum dimension R only.

Rank can be maximally R. Right. Because that is the number of rows or columns. Right. N is larger than R.

Right. So obviously each of these is maximal rank. That is what we proved. Right. Because their rank is at least R is what we are saying. But the point is each of these matrices can never have rank more than R.

Because they are deficient in the number of columns and rows. Yeah. So obviously each of these are maximally ranked. Now let us forget about this for now. But if you look at this guy, what does maximal rank mean for this? It means all rows are linearly independent.

Right. Maximal rank is only possible or rank R only possible if rows are linearly independent. And that is what was our theorem itself. Right. That is what was the lemma. If you have a relative degree R system, then these quantities are linearly independent.

Yeah. That is what we wanted to claim. So in order to prove this of course we use lemma, the 0.

1. Yeah. So this is where we wanted to get it. Yeah. In basically to show partial feedback in relation. Why? These form a new coordinates. And coordinates are too linearly independent.

Right. X and Y axis and Z axis, linearly independent. In fact orthogonal. Not necessary to be orthogonal but all coordinates have to be linearly independent. Yeah. Otherwise they are not coordinates.

Makes no sense to have dependent coordinates. So what is this? We define our new coordinates. Right. Because I already told you. Right. Feedback linearization, although the word says feedback, actually there is a state transformation also involved and a control transformation.

So there are two pieces to it. The state transformation is this guy. You take the coordinates as the output itself, its first derivative because notice LGH is 0. So obviously first derivative and so on and so forth all the way to R minus 1th derivative. Okay. And that is where the control will appear also.

We are already aware that the control will appear. And then you augment to it some functions. By the way this does not specify how to choose those functions. Feedback linearization does not specify how to choose the rest of the states. But remember if I started with n states, any valid new coordinate system also has to have n states.

Okay. Can't suddenly move from 6 dimension to 2 dimension. No. Right. The system is still involved in the 6 dimensions. So if you got feedback linearization only in the 2 dimensions, that is you had relative degree 2 system corresponding to your output, then you have to augment it with 4 more states which are what is called diffeomorphic maps. Basically these are maps that will make sure that this entire thing has a diffeomorphism between the old coordinates and the new coordinates. It is a one to one onto mapping.

One to one onto invertible smooth invertible mapping is a diffeomorphism. Yeah. Basically it means that I should be able to seamlessly move from one set of coordinates to the other set of coordinates. If I say this system is stable in this coordinates, it cannot be unstable in these coordinates.

Yeah. Everything has to be identical. Stable in this, controllable in this. Stabilizable in this, stabilizable in this, observable in this, observable in this. So that all properties are identical. Basically the equivalent of a similarity transformation in linear systems. Yeah. Similarity transformation in linear systems accomplished by diffeomorphism in nonlinear systems.

Okay. So that is how you choose the phi 1 to phi n minus r. That is the rest of the coordinates. There is no guideline for it. You have to choose it so that you get independent coordinates. The first are already independent.

Why? Because they are Jacobian which is this guy, is linearly independent. Yeah. Actually, yeah, I actually do not need the, let us say, I do not need the last one obviously.

Let us see. Yeah. Yeah. D H d l f r minus 1. So I actually do not think this should have been there. This should not be there and this should be r minus 1.

Yeah. Because that is the Jacobian. I do not know why I wrote it like this. This is not okay. This is actually d. Yeah. That would be the Jacobian basically. So how do you say that coordinates are independent? You basically take the Jacobian and if they are, if the Jacobian is invertible, then it is a diffeomorphism.

Okay. Local diffeomorphism. Okay. So this is the Jacobian, right, because you just take d d d everywhere. That becomes a Jacobian, right. And so this has to be full rank because this is an n by n matrix, remember. Right. And in order to facilitate this, we have already proven that the first r are already linearly independent.

We have already proven that this is rank r. Okay. We have already done our best is what we are saying. Rest you figure out yourself. You still have to figure out the rest of the coordinates.

That is on you, on us also of course not just on you. Alright. Alright. So basically that is what we are saying. These phi i's are n minus r smooth functions such that this entire map capital phi is a diffeomorphism and the Jacobian has full rank. It is not an and by the way diffeomorphism is equivalent to the Jacobian being full rank. Not an and. If you want to check that a nonlinear map between same number of variables to another same number of variables is a one one onto invertible smooth map.

You just have to take its Jacobian and verify its invertibility at a particular point. Obviously, it is a nonlinear map. So you can verify invertibility only at a particular point.

It is almost like linearization actually if you think about it. Yeah. Alright. So the anyway that is fine. I mean we do not actually need it but that is we will talk about that later. The system of course evolves evolution in the new set of coordinates is this guy which is basically going to look like this.

It is fine. It has some nice structure. We look at that rest of the piece slightly later. Alright. Thank you.