Nonlinear Control Design

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Week 1: Lecture 5 : Preliminaries: Normed linear space

You have seen the definition of norms. I already said that for norms to exist the space has to have certain properties right. Norms do not exist in all spaces. You cannot define a notion of distance in all sorts of spaces ok. But where you can is where we are working with. So we, so anyway, so I assume, I am sort of assuming that all of you have seen vector spaces in some form or the other.

Even if you have not seen for more often than not we are just working with Rn. Folks in physics and all they tend to work with complex Cn and all that. So but we primarily work with Rn. Sometimes we work with some more non-Euclidean spaces which are not vector spaces anymore.

But that is not part of this course. So we have courses running of course in those areas also in SYSCON. So we talk of non-linear space is essentially a linear vector space which has a norm ok. That is it. The norm, so just like we saw Rn with the infinity norm or the 2 norm or the 1 norm.

So these are all non-linear spaces ok. Why linear, so linear space and vector space are used almost identically. This is because vector space has linearity. So whenever we talk about linear structures in mathematics we are invariably talking about vector spaces of some kind ok. So therefore we say non-linear space or you can also call it norm vector space ok great.

Now we just want to make a quick proofs of why some of these norms satisfy the norm properties right. I mean very quick yeah we are not going to spend too much time on it ok. So let's look at the infinity norm first right. So infinity norm is essentially the max of the absolute value of the elements right. So obviously it is non-negative right because you took the absolute value and absolute values are non-negative done and if the infinity norm is 0 it means that the maximum absolute value is 0 and because we are talking absolute values therefore every other element also has to have 0 and absolute value yeah.

Therefore all it's essentially 0 infinity norm implies that the vector itself is a 0 vector ok great. This is obvious the scalar multiplication don't have to do much ok. So the only key thing in fact in most of these proofs the only thing that you really have to prove is the triangle inequality. So what is it how do we claim that we have the triangle inequality in this case? You look at the x plus y infinity norm and this is actually exactly equal to that by definition right. It is just max 1 to n of i x i plus y i ok.

This is exactly the definition of the infinity norm and I hope you understand that for some k it's exactly equal to this right. It's a finite length vector so obviously for some particular k it's exactly that value right the max goes away and this is obviously less than equal to this k because when I break the absolute values I get this inequality because this is actually using the triangle inequality for absolute values right yeah. And so this and of course I can say that this is less than equal to max of this plus max of this so I am done ok. Pretty simple nifty proof very simple yeah very standard yeah. Again works because it was finite dimensional so I could get from here to here yeah.

If we were talking infinite dimensional spaces big soup yeah not easy to prove ok. Alright great what about the two norm I cannot look at all the p norms so I am just going to look at the two norm which is the most popular norm yeah ok. First three properties very easy non-negativity obvious because it's not just absolute value I am even taking a square here. If each of them is zero then again the same deal yeah if two norm is zero then because this is there is nothing negative here so each term has to be zero yeah no two ways about it so it's a zero vector right and scalar multiplication is of course yeah too simple to talk about alright. What about the triangle inequality ok so here we need little bit of work ok.

So this triangle inequality so this is the two norm right and what does it look like it basically looks like this guy is again by definition exactly the definition for the two norm and then I expand it right ok I am just expanding it inside the bracket right then I am saying that this is exactly equal to this because this is actually non-negative alright that's true right this quantity inside is actually non-negative yeah it's actually x plus y square so therefore it is exactly this guy and now I am looking at the inner product of x y why am I trying to use the Cauchy Schwarz here. Why? We have to use the Cauchy Schwarz that's for sure unfortunately the equalities are not in the right place which is creating some confusion please don't mind that actually this is just this ok this equality is simply this ok this equality is just coming from here because it's x i square so I can simply write it as two norm of x square and y i square summation so I can just write it as two norm of y square yeah this is just breaking the summation no problem and then I am just writing this guy is exactly the I mean I have written it as it is because I can't do anything here yeah then I am looking at the inner product yeah which is this guy yeah by definition for R n yeah or you also then write it as the scalar dot product which is which is essentially this guy and this I know is less than equal to this ok. So I have not actually used the Cauchy Schwarz inequality itself I have arrived at it using this sort of a contrived method why am I doing it because I have not actually proved the Cauchy Schwarz inequality itself yeah the Cauchy Schwarz inequality is a more general inequality for general norm linear spaces so I have not proved it so I am sort of arriving at it in this sort of contorted way alright and so what do I have? I have that this particular piece is less than equal to twice norm x norm y and so this is actually just equal to this right just norm x square plus norm y square sorry norm x plus norm y whole square alright and then if I remove the square roots from both sides I have my tri-annual inequality alright. So by the way this proof is only for R n ok if you want to do this for general vector spaces the proof is a little bit more complicated and you can look it up it's available in a lot of places yeah but this we have done just for R n ok because otherwise this cosine theta and all doesn't make any sense what is cosine theta? What is theta even yeah? So if you have more general norm linear spaces yeah for example if your vector space is consisting of matrices then talking about this kind of cosine theta and all that is not very clear right in those cases you have to have a more generic proof yeah so that's also available I have just not talked about it here ok alright. We also talk a little bit about convergence because convergence is what we are trying to do in this course right we talk about asymptotic convergence in continuous time and all that but eventually the notions are very parallel to series sequence convergence that you have seen in your typical undergrad mathematics course yeah so it's just a extension of that in continuous time alright so the ideas remain similar yeah.

So we look at convergence and of course we want one of the important things is we also want vector spaces where convergence is well categorized yeah if convergence is not well defined then those vector spaces are not nice enough for you to work with yeah you can always create such funny vector spaces we will talk about yeah alright great. What is convergence of a sequence? All of you understand what is a sequence? It's just this, this notation essentially is a sequence it is just you can think of it as a function from integers to real numbers or vectors ok just a function of integers to vectors remember yeah so this I denotes that integer dependence yeah so there is a first term there is a second term there is a third term there is a fourth term so it's in a sense it's ordered yeah only in the integers not in the vectors itself yeah vectors could be anything. So sequence in the end and the most important thing is a sequence is always infinite length anything finite length is not a sequence yeah by definition a sequence has to have infinity many terms so all integers you know 1 to infinity have to be mapped yeah or whatever natural numbers positive integers have to be mapped ok. So sequence xi, i equal to 1 to infinity in this normed linear space is said to converge to some point x0 in this normed linear space if for all epsilon positive there exists a number n in positive integers such that xi is less than xi minus x0 is epsilon close for all i greater than or equal to n ok. So this is your first I mean not first but in this course this is your first introduction to epsilon and epsilon delta type definitions ok I am not sure if all of you or some of you have seen this before but this is how typical epsilon, epsilon delta, epsilon n definitions are made yeah the idea is very simple and because mathematicians want to capture it in a mathematical language which is precise this is required yeah otherwise the idea is very simple the idea is that how do you say there is convergence it means if you go far enough in the sequence how do you say that say that you are converging to some point x0 just a second if you go far enough in the sequence yeah you will reach to the you will be very close to this point x0 ok.

So maybe not at the starting of the sequence but if you go further further further further down you will start to get close to x0 yeah and it is consistent yeah it is not like the stock markets yeah if you once you get closer to x0 you are going to remain close to x0 it is not like you are going to just start bouncing back and go very very far away from it again no yeah once you get to this point say this far in the sequence beyond it everything will be very close to x0 yeah so this is exactly what is sort of characterized using this epsilon ok so the epsilon is a user given this is how we say it yeah depending on how the sequence of this statement is this epsilon is something that user given that is user given that is you give me an epsilon and I give you an integer n ok beyond which every term is epsilon close and how do we measure epsilon closeness norms that is exactly why we introduce norms that is the whole point of introducing norms ok. So this is how we make epsilon delta epsilon n definitions it is really capturing normal intuition in mathematical language as simple as that ok. Now what is an example of a convergent sequence ok xi equal to one over i this is a convergent sequence remember we are talking about sequence convergence not series convergence ok sequence is each term yeah this term is this term signifies a sequence so when do I say that the sequence converges it means that this term starts going towards the point series is completely different series we are talking about summations that is much harder to achieve by the way as you can imagine here I am just saying that the term goes to some value in a series convergence I have to say that the summations converge to some value ok much more difficult to achieve as you can imagine ok. So sequence convergence that is what I said converges to x0 here xi equal to one over i it is very obvious that it goes to zero right I can even find an epsilon I mean there is a computation here but not difficult how do I compute if you give me an epsilon yeah and I want my terms to be less than has to be epsilon close to the equilibrium in this case I know the equilibrium is zero right then I want each xi to be less than epsilon ok and when will each xi be less than epsilon I can compute that right then one over i is less than epsilon ok therefore i has to be greater than one over epsilon ok but one over epsilon is not necessarily an integer so I apply the ceiling function so one over epsilon is say hundred point three seven then the ceiling of one over epsilon will be one zero one the ceiling function yeah floor function is one zero zero ceiling function is just taking the higher value that's it ok so that's it this is how you compute if I ask you how to you give me an n this is how you compute bn alright great and you can see that it's very obvious that n has to depend on epsilon yeah n will depend on epsilon almost impossible to have a n which doesn't depend on epsilon right ok now one of the key problems with this kind of a definition is that you need to know where you converge yeah you need to know this x0 without this knowledge of x0 I cannot do any computation yeah and this may not always be the case and so there is this Cauchy sequence ok what is the Cauchy sequence exactly what you were saying a sequence is said to be Cauchy if for all epsilon positive there exists integer n such that xi minus xj is less than epsilon for all ij greater than equal to n ok so here we are no longer comparing with any particular point of convergence in the vector space ok here we are actually just comparing the individual elements right so what am I saying you give me an epsilon again ok and I give you a integer n such that if your i's and j's are greater than equal to this n and you compare them they are epsilon close this is exactly what is Cauchy sequence ok so there is a little bit of a problem but again this example in this case also everything works out fine it's also a Cauchy sequence yeah but there is a problem in the sense that convergence implies Cauchy but Cauchy sequence doesn't imply convergence necessarily ok I can always cheat and create these problem examples how so suppose I have my vector space and this is a valid vector space as this it is the 0 1 open interval 0 1 open interval is my vector space alright and I take my sequence as this guy ok xn is 1 minus 1 over n so where does it converge to as xn as n tends to infinity 1 but I cheated 1 is not in my vector space ok 1 is not in my vector space so this series is this sequence is not convergent as per my definition because the point of convergence has to be in x but this is a Cauchy sequence remember not difficult to see because as you as you increase n all your terms start bunching here on the right hand side so this sequence is definitely a Cauchy sequence I mean I am not verifying it formally but you can easily verify it formally that as n becomes I mean if you give me any epsilon I can give you an n such that your terms start getting epsilon close to each other yeah because everything starts bunching up here yeah you can start seeing that it gets very dense here in the corner ok so so sequence is Cauchy but not convergent because I created a funny vector space yeah and many such funny vector spaces you might encounter ok it's not difficult yeah if I give you a donut or if I give you a disc with no origin create all sorts of such weird vector spaces where your Cauchy and Cauchy sequence and convergence will not be equal yeah we don't like this so we define complete norm linear space or what is called a Banach space ok what is a complete norm linear space it is basically a norm linear space where every Cauchy sequence converges ok so the two notions are identical then it's a Banach space or a complete norm linear space ok so we usually work with Banach spaces yeah of course notions of Banach space and all that they are you will see them more prominently in PDE analysis and all that so of course but but notion is pretty straightforward the idea is Cauchy convergence sorry Cauchy sequence and convergence sequence notion have to be identical yeah so that is complete normed linear space or Banach space alright as you can imagine Rn with these norms is a Banach space alright because I excluded nothing so whatever sequence you create if it's convergent it is definitely Cauchy if it is Cauchy it's definitely convergent yeah alright similarly similar to normed space you have the inner product space also ok again for Rn these things are more or less become the same right so for an inner product space essentially you are saying it's a linear space with some inner product ok as simple as just like you said norm linear space linear space with the norm here you have a linear space with the inner product and similar you have these criteria for any function to satisfy an inner product so what is an inner product the inner product is actually a function which takes two elements in the vector space and gives you an element in the field ok f is the field alright so what is I mean simple examples basically field is what each component of the vector space is made of yeah in our simple cases it is Rn cross Rn and it maps to real numbers so inner product space you already know for typical vector spaces the typical vectors in fact it is just that scalar dot product right x dot y yeah but again matrices and all you have to think more carefully what this is yeah but still they are well defined yeah even for matrix matrix vector spaces ok so the properties are basically symmetry a distributivity the scalar multiplication property again and the fact that inner product of the vector with itself is non-negative and this property in fact the ras property should remind you very very much about the norms right right this very much looks like a norm property and this is why every inner product linear space is actually a norm linear space ok you can always use an inner product to create a norm yeah just like in real numbers in Rn yeah so you can actually see this just like in Rn what is the inner product this guy x transpose y or x dot y or whatever yeah this is the inner product and this immediately gives me a norm when I replace y with also x so basically when I take inner product of the vector with itself I get this norm which is actually what inner product of x with itself and a square root this is exactly the two norm ok so the inner the norm that you get from inner product is a particular norm it is not necessarily covering all the norms right obviously you can't get a infinity norm with this inner product yeah but it will give you one specific norm so in this particular case you get the two norm right so there is so that is not something you can control that depends on the inner product you had ok so just like you have a norm linear space and Banach space and all that similarly if you have a complete inner product space then it is called a Hilbert space just terminology ok you had a complete norms linear space it's a Banach space complete inner product space it's a Hilbert space we can we can assess completeness here also why because the inner product gives me a norm right and therefore I can check convergence and Cauchy convergence in that norm right as simple as that ok so you have Banach space Hilbert space so these are essentially the nice nice very very nice spaces that we always work in in fact not just in this course but also in PD control and so on if you do any course with Vivek you will see that there also you require these assumptions ok very difficult to work without these PDE or partial differential equation if you talking distributed parameter systems and partial differential equation when you're looking to do something like a boundary control on those systems there's something that my colleague Vivek specializes in and teaches a few courses on and there also you make these assumptions of course there the vector spaces are significantly more complicated their function spaces so infinite dimensional so notions are more complicated but what I am saying is that the same assumptions have to go through there also alright. Thank you. you