Nonlinear Control Design

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Week 1: Lecture 4 : Preliminaries and notations

Alright, hello everyone, welcome to course on Control of Non-linear Dynamical Systems. We are in the second day of our lectures, alright. So last time we, mostly we introduced the course and then we moved on to see a few examples, okay, of non-linear systems from, I would hope from a relatively wide range of areas, alright. You know, you had these disease spread models, you had tractor trailer models, so aeromechanical systems, atmospheric models and so on. We also saw some of the cases which are not very amenable for what we are going to do in this course, okay. And this has got to do with existence and uniqueness of solutions, right.

So we saw some examples of when you may have non-existence of solutions and non-uniqueness of solutions, alright. And we sort of, you know, got a feel for how this can sort of create problems for us in terms of the analysis. Now at the end of it we sort of made a blanket assumption, alright, that we are not going to consider these funny and extreme cases. So we are simply going to assume pretty nice smoothness like assumption, yeah, which is called the global Lipschitz condition, okay.

Which essentially looks something like this, right. I mean this is what the global Lipschitz condition says that the function of course is separately you require that the function be piecewise continuous in time, right. And on top of it you require something like a global Lipschitz condition. This is something like a differentiability but more than that, yeah, a little bit more than differentiability. And so this is more or less enough for solutions to exist and be unique for all times which is greater than initial time, alright.

So this is the assumption, blanket assumption we make going forward, okay, alright. So today we start with some basic preliminaries, okay, for, so you can see that I am using some of my own adaptive control nodes. So this material is common between the two. So we are simply going to look at some preliminary material. First we talk about a few myths and temptations, then we talk about more, you know, more basic things like norms, yeah.

A lot of you might already have exposure to norms and so on but we are going to still repeat it, yeah, just to establish the notation we are going to use, yeah. Most importantly this will basically set up the notation that we are going to use in this course. So later on it should not feel very alien to you, okay, alright, great. So the first sort of thing that we look at is a few myths that a lot of us carry when we do asymptotic analysis, okay. When we do any kind of analysis for that matter, alright.

So what are these myths and temptations? Let us look at this. First of all, if I tell you that there is a function which is real value, so it takes real numbers and outputs real numbers, yeah, and I tell you that the function converges to a constant, this does not necessarily imply that the derivative converges to zero, okay. So a lot of us sort of think that, oh, if the function is converging to a constant, the derivative should converge to zero, okay. It is, while it is very true that if the function is constant for all time, then its derivative is zero for all time. This is true, no doubt about this.

But here we are not talking about the function being constant for all time. All we are saying is the function is converging to a constant as t goes to infinity, yeah. Again I hope these notations are clear. If not, please go ahead and revise, yeah, your functions and continuity and limits and so on and so forth. We are not going to discuss this, okay.

We are saying that in the limit if the function converges to a constant, then the derivative does not necessarily converge to zero, okay. So some very easy examples, okay. So this is one example. So if you look at this function, sin of t square divided by t, okay, and if you take the limit as t goes to infinity, what happens to this function? The limit goes to zero, right? Yeah, everybody agrees because the numerator is just going to oscillate between minus one to plus one, yeah, and the denominator is basically going to go to infinity linearly, right? So therefore the ratio is definitely going to zero, yeah. This is, and the other thing to remember is that this is a pretty nice smooth function everywhere except, you know, at t equal to zero.

But since we are really talking about t going to infinity, we don't really care so much about t, what happens at t equal to zero. We are talking about large time rather than, you know, small time. So we are not that worried about its not so nice behavior at the origin, at t equal to zero, not origin, at t equal to zero, alright. Now let's look at the derivative, alright, very easy to compute, right? So I just use the, whatever, the, if you want to use the product rule or, yeah, you can use the product rule or you can use the ratio rule, yeah. The derivative is just minus sin t square over t square when I take the derivative of one over t and then I have to take the derivative of this guy which gives me twice t cos t square and so that cancels with the time, right? So I am just left with twice cosine t square, okay? I hope you are convinced that this is in fact correct, okay? Alright, now what happens? This guy still goes to zero, right? Again numerator continues to oscillate but the denominator is actually going to infinity, in fact much faster, yeah? But this term continues to oscillate, right? You don't know what it is doing at infinity.

In fact this function f dot of t does not have a limit as t goes to infinity, okay? This function does not have a limit as t goes to infinity. I hope this is clear, okay? Now can you give me another example? I gave you one example. Can you give me one more? It is very easy to construct with similar ideas I guess. What do you think? Cos t square by t. Making your life really easy there, alright, yeah.

Cosine t square by t, alright? So you can construct similar ones, yeah? You can also construct different ones, okay? I mean it is not too difficult, okay? It is not too difficult to do that, okay? Let's see. Yeah, yeah, yeah, you can construct different ones also, okay? I am not going to tell you what but yeah, so you can take cos t square by t as another example, okay? You can always do that, yeah? You can also do things like sine t cube by t square, yeah? So I mean so many different choices, okay? So there are so many functions. I mean large swaths of functions which don't satisfy this kind of a mistake. But I can promise you in a test I will definitely find one of you at least writing that because the function converges to constant, the derivative converges to zero, okay? This happens all the time, okay? So be very careful. I have, that's it.

Is it possible to write a kind of a theorem like this? Absolutely, absolutely. All of it, yes. All I am saying is remember this is a myth, yeah? I am not writing the test that my function converse to a constant so my derivative is going to zero, yeah? That's wrong, very wrong, alright? Okay. What about the converse? That's not true either. Yeah, the derivative of a function converges to zero, the function doesn't converge to a constant either.

So the converse is also not true, okay? So all of this mess is created because of the limits, yeah? So the limit makes all of these go wrong, yeah? Asymptotic. So what I am saying is asymptotically something happens to a function, doesn't mean that asymptotically the same, something similar nicely happens to the integral or the derivative, okay? So again an example, very easy, yeah? This is a much easier example. If I take the function as log t, log natural of t, then the derivative is what? One over t. What happens? Goes to zero, yeah? The derivative is going to zero, yeah? Right? If I take the limit of the derivative going to zero as t goes to infinity, right? Just one over t, right? What happens to this guy as t goes to infinity? Blows up, right? Really bad, yeah? Blows up in fact, yeah? So sublinear growth, it's a sublinear growth but it is still, it's going to go to infinity as t goes to infinity, okay? So really bad functions, okay, in that sense. Again another example, can't use this one, difficult now to continue this one because one over t square will not work.

Will one over t square work? One over t square, if f dot of t is one over t square, then fine, it goes to z, the derivative, f dot of t is going to zero. What about the integral of one over t square? Zero goes to zero, okay? It's a well-known fact actually, yeah? I mean when you have a, when the denominator has powers more than one, strictly more than one, then this is a convergent series, yeah? I think those of you who have seen series, they would know that if I take a series one over n, yeah, then it's not necessarily convergent but one over n square, it is, yeah? So the similar idea actually comes through here also, okay? So this sort of an example will not work, something else? Motivated by the previous example maybe? From the previous, like this one, motivated by this, can you construct something? And not the one that I crossed out, this is wrong, or any other counter example where the derivative converges to zero but the function itself does not converge to a constant. t e minus t, so the function, the function is going to zero as t goes to zero. e raise to minus t will dominate t, right? You are saying, yeah? Yeah. Exponential will dominate the linear growth, so e to the power minus t will take you to zero.

Square root of t, nice example, alright? Square root of t, what happens? F of t square root of t, F dot of t is, did I get this right? Definitely did not get this, this is minus, right? Minus one over two x square root of t, the derivative, right? So now this guy is going to go to zero as t goes to infinity, right? But this is again blowing up. This is going to infinity, yeah, as t goes to infinity, right? Is okay, right? Yeah, yeah, okay? So anything with square root of t, sin of square root of t, this will also work, right? Same examples, alright? So many examples, again, I mean, no dearth of examples here, yeah? So, please resist from ever saying anywhere that if functions converge to constant, the derivatives converge to zero, derivatives converge to zero, functions converge to constant and things like this, okay? So, we very vary because we do asymptotic analysis all the time and I can tell you this is a serious temptation. While you are writing things in the flow, you tend to write it, yeah, okay? I am done, yeah? But that is not, okay? You have to prove those things, that it really happens, that the function converges to zero and the derivative also converges to zero or something like that, you have to actually prove it, yeah? It requires something more, yeah? Alright, alright. We will probably at some stage also talk about what is this something more, but not immediately, alright? So, now we go on to a little bit of more pedantic material, yeah? I don't know how interesting you will find it, but anyway, I mean this is like I said, the purpose is to establish notation and we will establish it, alright? So, the first thing is vector and matrix norms, right? So, we keep using norms all the time because this is more often than not we will be working with real vector space, right? So, everything is real numbers or Rn or Rk and Rp. So, we want to have a means of talking about distance, lengths of vectors and so on, yeah? So, how do I say that my state is zero? I can only say that in the norm sense, right? So, or if I, how do I say that the state is getting closer to zero? I mean staying zero is still easy if all the components are zero, but how do I say that my states are coming close to zero, yeah? It has to be in a norm sense, yeah? So, because this is the only way I can measure some distance, length and so on, okay? So, we are always working with what is a norm linear space, alright? So, this is the idea, we will maybe talk a little bit more detail on this, but for now, remember that we are always working with a norm linear space, yeah? There are several notions connected to it, again we will look at this in a little bit more detail, yeah? So, so this bit will be a little bit pedantic and mathematical.

So, norm is basically a function which takes your element in the vector space, in this case we are just saying Rn, yeah? But norm is valid, I mean you can generate a norm for any norm linear space or any norm vector space. So, this function is a valid norm if it satisfies these properties, okay? Very standard properties, non-negativity, yeah? And then you have the scalar multiplication property and triangle inequality, okay? So, these are really the three properties and of course, this is a key addendum in some sense, right? If the norm is zero, the state or basically x has to be zero, okay? So, there is no two ways about it, alright? So, what are the typical commonly used norms for vectors? It's the infinity norm and the p norm. What is the infinity norm? It is just the largest element of the vector, largest absolute value of the vector elements, okay? So, you take the absolute value, take the max, okay? And what is the p norm? The p norm is basically just take the absolute value to the power p and then take the summation and then take the pth root, okay? So, the most common one amongst these is the, what is the Euclidean distance, right? So, if I have, for example, if I have a vector in R4 like this, right? Like this guy, then what is the infinity norm? It's 7, right? Because if I take the absolute value of all elements, 7 is the largest one, okay? Similarly, what is the one norm? It's just the, you know, the one norm is just basically, you know, it's just the sum of all the elements in this case, right? Again, because all the elements are positive, right? So, the one norm is just the summation. So, that's 3 plus 2 plus 7 plus 5, that's this guy, right? And what's the two norm? Is this expression, right? Take the squares, basically take the squares of the absolute values, take the summation and then take the pth root of this, okay? So, this is the standard way of computing distance, this is the Euclidean distance. We are used to computing distance with the two norm always, yeah.

The other norms we don't, let's see, I'll not go there first. The other norm we don't use like, you know, immediately, depends on, those are very special purpose norms, we don't always use them. So, I will say that it's, some of them are little bit more mathematical constructs, but they are all still very very useful, okay? So the one norm is of course a very very important norm. Alright, now one of the questions I typically ask is, so suppose I go here to get some space, suppose I tell you that X is in R2, okay? Two dimensional vector, okay? So this is the notation, X belongs to R2, okay? I hope you are used to this notation.

Alright. So, now I ask you, what does the set A1, which is basically norm of X infinity less than equal to 1. What does this set look like? Square. Square, okay. Alright, so you were saying, I think I can draw some lines still, let's see. Okay, so what you are saying, it's a square, centered at? Centered at origin, okay, alright.

So it's a square centered at origin and how big is this square? Right. One, one unit each, okay? So I don't know if I made a square. Alright, this is the square, okay? So right, you are right. So this is basically a square of, basically this is 1, 0, this is minus 1, 0 and this is 0, 1 and this is 0, minus 1, okay? Why is this a square? Can anybody sort of understand, figure why this is a square? Why? Why, why? Distance of what? So the infinity norm is what? Is the maximum in x and y, alright? Okay, then now what? Exactly. Exactly, yeah, so basically, since the infinity norm is less than equal to 1, it means each component in absolute value is allowed to be less than equal to 1, right? And so you can see that all of those are right here in this square, okay? You basically, if you get out of this square, you are guaranteed to violate this condition, okay? As soon as you get out of this square, you are guaranteed to violate this condition, okay? So that's the whole idea, as simple as that, okay? Great, great, great, okay, good.

Now what about A2 which is, right, this everybody knows very well, right? Okay, I can draw a circle, I have assistance, alright. So now I just have to make some centered lines here, alright, it is what it is. And the radius is? 1, alright? So circle is basically the equation of the circle is x square plus y square less than equal to 1 in this case, so the disc, not the circle but the disc. So and that's exactly what we are doing, alright, great. Okay, this was actually wrong, this should have been called A infinity, right? What is A1? 1 norm less than equal to 1.

It's two-dimensional, x is two-dimensional, it's a diamond, yeah, x is two-dimensional. Yes, it would be a line if it was one-dimensional, yeah? So let's see, let me try to make this one. Yeah? Why? Okay, so basically you will get, if you take the absolute value, right, so what you want is that the equation that you get here will be absolute value x1 plus absolute value x2 less than equal to 1. This breaks down to four lines, right? x1 plus x2 less than equal to 1, x1 minus x2 less than equal to 1, minus x1 plus x2 less than equal to 1, minus x1 minus x2 less than equal to 1, okay? So therefore you will get the area as the intersection of these four, right? So the line, basically the end, the rhombus edges are the lines, the four straight lines, right? Like this line is, what is this line? x1 plus x2 equal to 1, sorry if you say so. Yeah, I guess so, this one should be x1 minus x2 equal to 1, then so on and so forth, yeah? It is just the intersection of these lines and whatever is contained inside them because you have a less than equal to 1, yeah? If it was greater than equal to 1 then it would be everything outside, okay? Good, clearly you guys have seen all this before, alright. Thank you.