Nonlinear Control Design

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Week 7 : Lecture 39 : Passivity based control: Part 2

So, passivity, zero state observability for this system x dot is f x u y equal to h of x. We are going to use this immediately to design stabilizing control I mean that is the theorem that is this theorem. What does it say? It says that if you have if the system that we just talked about is passive with a radially unbounded V of x no longer just V semi definite positive. We are saying that the V is radially unbounded because obviously we want to use Lyapunov theorem. So, if the system that we just saw is radially unbounded is passive with a radially unbounded storage function V of x and zero state observable then any feedback of this form where phi is locally Lipschitz in y, phi is zero at zero and y transpose phi y is strictly positive for all non-zero y. This globally asymptotically stabilizes the zero state.

So, how do you construct the feedback? You basically declare the control to be any minus phi y with these properties Lipschitz zero at zero and actually positive you know y transpose phi y is positive definite that is its you know for non-zero y it is strictly positive. So, very nice result actually I mean and really really useful. So, let us just look at what happens. We look at the proof.

In this case we will look at the proof just to see sort of what happens. So, this V that we have we are already assuming that for this system you have a radially unbounded V. So, we take this as the candidate Lyapunov function because it is already C 1 and already radially unbounded. So, pretty strong properties there and we take this as the candidate Lyapunov function for the closed loop system. So, the system now becomes this because I have plugged in my feedback because it is minus instead of control I have minus phi y because that is the feedback I have chosen and then the output is just h of x.

Further what do we know about this V of x? We know that the derivative this is the passivity property itself that partial of V with respect to x multiplied by this guy is actually less than equal to u transpose y. And the funny thing is u is just minus phi y. So, I plug this guy in and by this assumption of positive definiteness what do I have that this guy is negative semi definite not negative definite ok. Remember why because why is it not negative definite actually I should not say because you tell me why did I say this is only semi definite and not negative definite. The same thing I say every time almost in every class why is this V dot only negative semi definite what did I do? I took the partial of V multiplied by absolutely thank you very much.

It does not have the entire state x it only has y which is less. So, if you do not have all the states cannot be definite at all by our assumption it is sign definite, but not positive definite it is only negative semi definite alright excellent good good good good. Now, we are ready to apply the Barba-Sheem-Krasovski-Lassalle theorem ok. Now, you remember why this zero state observability looks very much like the Lassalle invariance condition right because of this we are going to invoke that because we have we started with the ready unbounded V, but we ended up with a only negative semi definite V dot right. And in all such situations we invoke in some version of Lassalle's yeah whether it is a original Lassalle invariance or it is Barba-Sheem-Krasovski-Lassalle.

In this case we are looking at the stability of the origin. So, it makes sense to use the Barba-Sheem-Krasovski-Lassalle we are not dealing with limit cycles and all that complicated stuff we are talking about stability of the origin. So, no need for the original Lassalle no need for the omega and all that all we need to do is have a ready unbounded V excellent we do have a negative semi definite V dot we do next step construct the set V dot equal to zero and that is exactly this guy phi transpose y equal to zero that is guy this yeah and this is I am claiming that this is actually equal to y equal to h of x equal to zero y y is this equal to this absolutely for any non zero y this is strictly positive. So, if this is actually zero the only way that is possible is if y is actually equal to zero that is why these two are the same just by our assumption nothing bigger than that excellent and so by zero state observability because this is now that set that we discussed in the zero state observability exactly. So, in fact, it almost look like we have defined things and assume things.

So, that everything works out yeah alright, but these tend to work out a lot of times it is quite amazing you will see some examples yeah. So, a lot of times these assumptions do get satisfied is what I am saying it when I am stating the theorem it looks like I made some ridiculous looking assumptions just to make sure that my theory gives me a stabilizing controller, but that is not the case you will see it works also at times anyway let us look at this. So, this is the zero state observable set what do we know we know that no solution, but the trivial solution will stay in this set when control is zero when control is zero remember. So, let us look carefully what happens in this set y is equal to zero right which means the control which is phi of y minus phi of y is actually minus phi of zero right and that is zero. So, on this set control is zero which means we are already looking at the uncontrolled system and we already know by zero state observability assumption that for the uncontrolled system only the trivial trajectory is inside here nothing, but the trivial trajectory.

So, we are done this is what we need in barber sheen krasovskiy lassal right that the largest invariant set inside E has to be nothing, but the zero trajectory. So, you just completed the arguments all right is that clear pretty straight forward proof actually make sense all right. I do not know what I have written on the sides here let us not worry about this yeah you do not have to worry about this that is fine we will look at that later all right. Now, I already said that it looks like we made up these assumptions. So, that things go through and all that and it is true to some extent, but there are ways to make sure that this goes through these assumptions are getting satisfied.

One of the methods is passivity by output selection in a lot of systems that we are designing controllers for there may not be real outputs we might have the flexibility to choose outputs. So, that you have the passivity property with respect to that output remember passivity is a property which is somehow related to input and output. So, but if we have the choice of choosing an output you might be in good shape. So, that is sort of one of the things right I mean I think that is what is an example I am trying to give here in a very messy handwriting unfortunately. For example, if you just an example if I take this system this is the standard integrator type system we will be looking at in back stepping.

So, I hope you recognize this system already very quickly yeah in back stepping you would take x 2 is minus x 1 then you will take clf as x 1 plus x 2 plus x 1 squared and so on and so forth yeah we are used to that. For passivity what I am going to do is I am not trying to construct you know a clf or anything as of now right because we want to use barber shinkresovski type ideas. So, let us see I will choose my measurement as this position or the x 1 state if you think of mechanical system it is the position state I just measure one state x 1 what happens so, the question is can you choose a v appropriately and this is what we have to sort of you know think about carefully I am not sure if I have actually given a v choice here I want this v dot to be less than equal to minus less than equal to u transpose y and that is anyway in this case it looks like I did not actually make a choice proper I think actually let me come back to this later I did not make this choice properly yet this is not a complete problem here I will get back to this later. But the point is if you have such a system there are ways to sort of use output selection like this to make it passive that is the whole idea I am I this is not a completely worked out example so, I am not going to look at this right now let us not worry about it. But let us look at the theory first and see what happens because there is already another nice example here right if you have this kind of a system right which is again a control affine system and as usual you have some state some control Lipschitz in x and you further suppose you have that your drift system is stable not asymptotically stable but stable what do I mean by that that partial of v with respect to x times f of x is less than equal to 0 is actually I mean this is basically stable unforced system or it is basically you are saying that without the control the system is at least stable it is not asymptotically stable not going to converge to the origin or anything but it is at least stable it is not going to escape or anything great.

So, if you have that then you can choose your y as this okay to make the system passive okay do you understand why because of this yeah if I take the same radially unbounded function v okay and I take its v dot along the entire trajectory not just the uncontrolled trajectory the entire trajectory I get this yes it is just del v del x fx del v del x g xu okay and now I am saying that I want this to be less than equal to y transpose u okay so I am basically artificially choosing this as y itself okay why how did I get from here to here the first term is less than equal to 0 right so I can pretty much forget this guy okay therefore I know that this is less than equal to this much okay now if I choose my y transpose as the first piece then I am done right I have my passivity property okay this artificial looking property it is just by choosing an appropriate output I have this passivity property okay again might seem artificial to you but the point is if you have the freedom of choosing the output and your aim is just doing control design it is not this y I do not want you to think of it right now as the actual measurement from sensors and things like that you just think of it as a tool to design your controller okay once you have designed your controller you can figure out how to implement it and all that is a later matter but right now you are just using this y even if it looks artificial it is just a way of designing a control of it alright okay great now so we know that this sort of a choice for y will make the system passive okay good so what is the example this sort of example okay again not too far-fetched not too far from this way I have just given a particular form for the drift term here that is all yeah it is not too far-fetched from what we already have yeah now let us look at this V of x which is why did I choose this form is because this is actually making the system stable okay so you are trying to make the system stable also yeah suppose I choose my V as x 1 to the power 4 over 4 plus x 2 squared by 2 okay why did I choose instead of square in both fourth power in one and square in the other anybody because there is an x cubed just to cancel this x cubed term yeah if you take the derivative what happens yeah you forget the control and you take the derivative it comes out to be 0 exactly 0 yeah that is why I chose the x 1 to the power 4 instead of x 1 squared just to cancel this x 1 cubed term okay make sense is how we keep manipulating early up and off candidates this is a pretty good idea okay great so V dot turned out to be 0 and V was radially unbounded what does it mean the Danforth system is stable or uniformly stable in the sense of Lyapunov okay great now I want to make the system passive right now what do I need for the system to be passive I need this to be less than equal to U transpose y yeah but in this case the right hand side was 0 alright so I can I am free to choose any y actually because the right hand side is pretty much 0 V dot turned out to be 0 so V dot less than equal to U transpose y means I can choose pretty much you know wait wait wait did I get this correct V dot is 0 plus x 2 times U okay yeah that is fine so that is fine this is stable so basically what am I choosing as my y I will just go back to this formula I think it is better that I go back to the formula what does this formula give me in this case what is g of x yeah there is no g of x just identity 1 okay and what is so again I have to be careful g of x is not 1 what is g of x actually g of x for this example is 0 1 yeah it is a second order system so we have to be a little bit careful okay what is partial of V with respect to x yeah partial of V with respect to x is this in fact whatever I mean it depends on how you want to look at it but typically I take gradients as row vectors so what is this formula give me it gives me what it just gives you y equal to x 2 right partial of V with respect to x multiplied by partial of V with respect to x multiplied by g of x is just x 2 right so what our claim is that the system is passive with y equal to x 2 okay on top of this in fact it turns out that in this case my system is also 0 state observable with this y how do you have how do you claim 0 state observability unfortunately I cannot pull it up but yeah how do you claim 0 state observability what do you need to check yeah in the set h of x equal to 0 only the 0 trajectory exists so here what is it what is the set h of x equal to 0 it is x 2 equal to 0 okay and then I am sort of invoking lassal invariance type ideas yeah similar idea right if x 2 has to be 0 what is the largest invariant set inside x 2 equal to 0 set x 2 dot also has to be 0 that is how we do it right and if x 2 dot has to be 0 what is my dynamics x 1 dot equal to x 2 sorry x 2 dot equal to 0 right I just proved x 2 dot equal to 0 I need x 2 dot equal to 0 so if x 2 dot equal to 0 if you look at the dynamics of x 2 without the control because we are talking about the uncontrolled system solution we are not talking about for the uncontrolled system if x 2 dot has to be 0 then x 1 also has to be 0 that is the only way yeah because if x 1 is non-zero and anyway there is no control then x 2 will move away right from the 0 value therefore you cannot stay in this set y equal to 0 okay so the only way this can happen is if x 1 is also 0 therefore we have just shown that the largest invariant set inside y equal to 0 is both x 1 and x 2 equal to 0 alright make sense alright so this y is not just giving us passivity but also zero state observability so you know pretty much immediately that I can apply this theorem okay I can apply this theorem to construct a asymptotically stabilizing feedback not just stable the system has stability but if you want asymptotic stability you can immediately use this okay essentially you just need a function of y what is in the what is it in this case is just a function of x 2 yeah so that is the cool thing interesting thing if you may that you only need a function of x 2 in the control you do not even need the first state yeah so if you were a control engineer or a practitioner you can pretty much say that I only need to measure velocities to implement a controller for this system because it is only a function of x 2 because I just so basically if you think about it what would I choose as my control my requirement for the controller that phi is Lipschitz phi 0 is 0 and y transpose phi y is strictly positive for all non zero y okay and in this case we have chosen y is x 2 okay so what is it I just choose my control as this guy just minus k x 2 for example yeah I know that this is at 0 value of output control is 0 right I also know that y transpose phi y is basically just k x 2 squared in fact yeah okay so therefore it is strictly positive I mean it is 0 only when the state is 0 okay therefore this is a valid control and that and that is fine I mean it is just looks like a in this becomes the control law yeah as a function of the state x 2 this is the controller okay but I can do even better I can actually design a saturated controller which is again something that lot of engineers care about that the control does not have large magnitudes so all I need to do is I need to satisfy these properties right so what will I do instead of choosing k x 2 I take k tan hyperbolic x 2 what is the tan hyperbolic function do it takes any argument and it fits within plus minus 1 it is a saturation function yeah it is a smooth saturation function you can also have non smooth saturation function like you know like this can be non smooth saturation function but this is sort of a smooth saturation function okay so instead of taking minus k x 2 I can take minus k tan hyperbolic x 2 alright and this is a very nice function it is 0 only at 0 so therefore it satisfies this property also right it is actually it maps exactly like if it is if your y is like this then this becomes sorry like this yeah so it never yeah it is like this yeah it when y is positive tan hyperbolic y is positive when y is negative tan hyperbolic y is negative it is 0 only at 0 it is a actually a sign a mapping which maintains the sign also so it is a very nice mapping just saturating it okay so minus k tan hyperbolic x 2 is also a saturated control choice that you can do and in that case your control is just lying between these two yeah this it is flipped just because of the negative sign that is all okay so if you have the ability to choose an output yeah which will make the system passive and also zero state observable then you can directly apply this result okay so in this case also for a system like this also for a very general case like this you will have to think it will not see in this case what did we have we sort of assumed that your f of x is a stable system gives you a stable system okay in this case that is not evident that that this system is going to be a stable system or not okay if it so happens that this system turns out to be a stable system in the sense of Lyapunov then of course you can apply the same result alright otherwise you have to figure out how to play with these terms okay and that is what we will see in the next sort of trick if I may okay. Thank you.