Nonlinear Control Design

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Week 7 : Lecture 38 : Passivity based control: Part 1

So, weird condition, one is the passivity condition already unusual and then there is the zero state observability condition which is not too unusual. Let me see if I can try to sort of justify this. Let me see, I am not sure if I can. Let us go to the sort of linear system case. And also we will test this condition in example. So, you will anyway you know figure it out how to use it.

It is almost like this LaSalle invariance condition. It looks like that if you see. You have a set which is some function equal to zero and you are saying it does not contain anything but the trivial solution that is the zero solution. So, and what is it a solution of? It is a solution of this system, the uncontrolled dynamics.

Whenever you put the control to zero you are saying it is the uncontrolled dynamics. No solution of this uncontrolled dynamics is here except the equilibrium, zero solution. Equilibrium itself is a solution. It is a trajectory. Well, not this but this still a trajectory.

Is that clear? Great. Now, let me try to sort of at least make sense of it and try to connect it with our linear system idea. This is our uncontrolled linear system LTI system and our output typical. Of course, we assume things like you know x in R n, y in R p and all that. Same thing.

Typically p will be less than n, less measurements than states. Great. Now, let us see how do you write the solution in this case? It is x t is e to the power a t x 0. Correct? Just the using the exponential. So, this will give me the output is c e to the power a t x 0.

Right? So, how do we, why do we have the observability matrix condition? Because if I expand this, what is it? Identity plus a t plus a squared by 2 factorial t squared and so on multiplying x 0. Right? And of course, you can already start to see. I can write this, actually I can write this as an infinite series if you want. This is c c a c a squared multiplied by some something here. Yes? Yeah? And what you have here is the controllability matrix, sorry observability matrix.

I does not have to be infinite length because of I can make the infinite length into finite length y. This guy? Do not need to check this infinite matrix and only need to check the finite matrix y. Just two words. Name of a person, a theorem, y. Why? Can I shrink this infinite? Because this is infinite.

Right? I hope you believe that this is an infinite series. So, obviously it is an infinite length matrix, but your typical controllability of typical observability matrix is only till what? c c a squared until c a n minus 1. Why? Just due to Cayley-Hamilton theorem. Because all the higher powers are anyway you can write them as smaller powers. So, no need to write all the powers.

So, this is the observability condition. Alright? Great. Now, why this? So, let us see what is the set, this set in this case. What is the set in this case? It is in the linear case I am saying that x in R n, h x is equal to c x is equal to 0. Okay? So, if I use my solution this is actually equal to the set of x 0 in R n such that let us see c c a c a n minus 1.

Okay? Because everything else is independent of states. Right? All this quantity, this is all independent of states. It contains what? It contains time and this matrix. Right? Because once I use Cayley-Hamilton I will just have some complicated functions of time here. Is that fine? Just by the way each of these will be infinite series inside this.

But we do not care about all that. The idea is this. In fact, I do not think this is writable like this. We have to write it in a different way. This is because this is not compatible operation anymore.

x 0 is in R n and this is the, what is the dimension of the Gramian? Sorry, the observability matrix? Number of rows is kya bata? Number of rows is p. Number of columns is n. It is a p cross n matrix. It is a p cross n matrix. I think we will have to write this not like this.

I apologize. I am going to erase this for those who copied already. This is just doing vector math. I am not doing anything magical. This is x 0 transpose c c a c a n minus 1 transpose times this vector equal to 0. You will get something like this.

You will get something like this. Why I just did if you notice this is now compatible. Compatibility is a huge result. Why? This is a 1 by n.

This is n by p. And this is whatever. This is whatever dimension. This is of course appropriate to make it a scalar. So this is fine. This is now compatible and you can see that this is now, this is the set.

Now tell me something. What is the set of x 0s that will make this 0? By the way, this is to be for all t and all. t cannot play a spoilsport here because when I say that the set is 0, it has to be 0. t, there can be no t and all happening here. It has to be for all t.

t cannot mess it up. So I am looking at set of all initial conditions such that this product is 0. How did we get this product? Just from here plugging in the solution for x. Just from here we get this. Just plugging in the solution for x. Nothing magical and expanding the exponential and rewriting it in an appropriate matrix product form.

That's it. I mean even if you are not convinced about this, you can go back and think about it. That's not a problem. It will come out to be like this. Now tell me something. When will this be 0? For what x 0s will this be 0? Anybody? Yes.

Good call. For x 0 equal to 0, yes. Can you give me a slightly more complicated difficult answer like is that the only choice of x 0 for which this will be 0? See because this can do nothing, right? Like you can pretty much forget about this because this is all a function of time and all this 0 thing has to hold for all time. So obviously this product has to become 0. This product has to become 0. When will this product become 0? Yeah, yeah, somebody was saying something.

Whoever said that word? Null space. Null space whenever x 0 is in the null space of this matrix. What is the null space of this matrix called? I think you guys have either not done this linear system very well or you have forgotten. Unobservable subspace. The null space of this matrix is called the unobservable subspace.

So, alright. No problem. And this is called the unobservable subspace. So whenever x 0 belongs to the kernel of the observability matrix or basically x 0 belongs to the unobservable subspace, only for those x 0 this is 0. Okay? Okay? Alright? And once you can find any one x 0 like that you have created a trajectory, right? Because yeah you understand, right? As soon as you give an initial condition I have solved this, I obtained a trajectory, right? So for every x 0 in unobservable subspace I get one such trajectory, right? Now if you say there is nothing but the trivial solution, what are you saying then? You are saying that there is no x 0 in the null space other than 0 itself. Okay? So null space is empty is how we say it.

Zero we do not count. Yeah? So we are saying the null space of the observability matrix is empty which means system is observable. Okay? System is observable. So what we have just codified in this slightly more complicated language is just the fact that system is observable. Okay? What we are calling this zero state observability is actually the observability condition that you have for linear systems. At least for linear systems it boils down to that.

For non-linear systems you can have slightly more complicated notions of everything. Yeah? Which is why it is called zero state observable. Yeah? So basically you are saying that if you write this condition out for linear time invariant systems all you are saying is that the system is observable. That is it. Okay? Alright? Make sense? No confusion? Alright.

This? Ah. Okay. Okay. Your question is why do we say that the system is unobservable? Okay? Why do we say that the I defined observability as being able to identify the initial conditions from the outputs? Okay? That was the idea. Now if you look at this expression yeah or I mean basically this guy is just this guy. If you look at this expression the question is can I reconstruct x zero from y? That is the question I am asking. Okay? Now if the system is observable let us look at the good case. System is observable means this is maximal rank.

Observability means it is maximal rank. That is only then the kernel will be zero. Right? So what is the maximal rank? P. Right? Because it is an P by N matrix. Wait a second. Did I get this correct? Is it a P by N matrix or not? C is P by N.

C is P by N. This is also P by N. This is not a P by N matrix. Ridiculous. I was wrong. You guys did not correct me. This is a this matrix is a what? P by N times P matrix.

Is that correct or not? See this C itself is a P by N matrix. Right? So this is also P by N matrix. Right? Wait a second.

Wait a second. Wait a second. C is P by N. First of all this is all messed up. Anyway that is fine. It is supposed to be written in that way.

C is P by N. This guy is also P by N. Everything is P by N. And how many such entries do I have? But this seems wrong to me. It is very wrong. Yeah.

Actually this is not the observability matrix itself. This transpose is the observability matrix. This is the observability matrix itself actually. So it is C CA. That is why it is all coming out to be messy in my head.

Yeah. Because actually I am sorry it is just vector arranging the vector in matrix multiplication. Nothing magical. This product like I said can be written as this guy. And this is the observability matrix.

The transpose with the transpose. So now the dimensions are okay. What is the dimension of this? It is what? P by N.

NP cross N. Right? NP cross N. Thank God. It is becoming NP cross N. Did I get this right? NP cross N. Then this product is not compatible. I did not get this right then. It is some function of T times C CA CA N minus 1 times X 0 equal to 0.

Okay. This is now correct. Yeah. Because this was an NP cross N matrix. The observability matrix is an NP cross N matrix and the X 0 is an N by 1 vector.

This is now correct. Okay. Alright. Big mess. But the basic point is this expression here can be written as this. Okay. Now this can be written as this. Yeah.

And we are saying that this guy is full rank or maximal rank. What is the maximum rank of this guy? N. Yeah. Because P is of course greater than equal to 1.

So it is N. So it will be an N ranked matrix. Okay. Which means what? Which means what? So basically you have some N ranked matrix multiplying X 0. It is a solvable system of linear equations. It is a solvable system of linear equations. Okay. The simple, if you want to think more simply assume P is 1 single output.

It is a single output system. Okay. If it is a single output system then this is an N by N matrix. So basically what you have is a N by N matrix multiplying X 0. Which means it is an invertible matrix.

Right. So I can invert it and get my initial condition. Okay. So that is the whole idea. I need to be able to compute the initial condition from the given data. What is the data? Data is the measurements. That is observability.

Therefore if you have anything in the unobservable subspace. Yeah. So these are all funny. It is a funny thing. See just think about this. Okay.

If I have multiple points in the unobservable subspace X 0 1 and X 0 2. Okay. For both of them this is equal to 0. Correct. For both initial conditions my measurement was 0.

For both initial conditions my measurements along the trajectory are 0. So I cannot distinguish the two initial conditions anymore. Okay. Which is why this condition.

There can be no trajectory but the trivial trajectory in this set. Okay. So anyway this is itself a very nice subtle topic. It is not that easy to follow. Which is why and also I am not teaching this.

It is more haphazard. So I was just trying to make a comparison. Yeah. In linear systems the idea of observable unobservable controllable uncontrollable this in itself is a relatively involved topic. I mean you have to sort of wrap your head around it. Takes a little bit of time. Okay. The idea is if you have unobservable subspaces then you will have initial conditions which are indistinguishable by measurements.

You can take you know measurements and you will not be able to distinguish. Okay. Between the two initial conditions. Okay. So I mean you can even say simply something like let me see. If you have y equal to c e a t x 0 1 and also equal to c e a t x 0 2.

These are the same. This is the measurement. Right. And two different initial conditions. I have the same measurements. Then basically these two are indistinguishable. Okay. And if these are indistinguishable what did I just prove that c e a t notice x 0 1 and x 0 2 are different.

I just proved that x 0 1 minus x 0 2 is in the null space and x 0 1 minus x 0 2 is non zero.

Not a trivial vector. So it is in the null space of this.

Which means observability matrix is not maximal rank. So you can go both ways. Yeah. And you know sort of understand that you observability that is being able to construct initial conditions from measurements requires that observability gramian or observability matrix be maximal rank. Okay. And that condition is what for non-linear systems is codified in this sort of zero state observable idea. Yeah. Again why it has this funny name is because observability has multiple there are multiple observability and controllability notions in non-linear systems.

Okay. So more complicated. Okay. So that is the only reason why we have these multiple notions. Any questions? So two important definitions. Passivity. This we have not sort of connected to linear system.

We might later on as of now no need to connect it. It is just a general notion or a general property of systems and the zero state observability matrix.