Nonlinear Control Design

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Week 7 : Lecture 35 : Backstepping method for control design: Part 2

Great, so this is what we want to do. We want to try this out as a control Lyapunov function for the new system, ok. And like I said, the purpose of backstepping is to come up with a control Lyapunov function. Everything else is too easy, right. After that you know what to do, ok. Great, so this was the claim, so I want to prove this claim.

How do I prove this claim? Basically, yeah, I mean just take derivatives and so on and so forth, ok. So you will usually do this, you know, Lfv and Lgv and all that. But you know the simpler way. Just take the directional derivative and whatever term is multiplying the control is the Lgv and other term is the drift term, ok, Lfv, ok.

So I will actually compute, sorry, v dot x comma xi and this is what? This is partial of v0 with respect to x. So I am taking, I am just taking the derivative piece by piece, yeah. So I get the partial of v0 with respect to x and then I get what? Fx plus gx xi which is x dot, right. And then there is no partial of v0 with respect to xi because v0 is independent of xi. So done, so there is no partial with respect to xi.

Now I take the partial of the second term, right. So this will give me xi minus k0 x transpose, yeah. So this is, I am assuming this is the Euclidean norm, the 2-norm. So this is just xi minus k0 transpose xi minus k0, right. So I am just taking the partial just like I would take the, you know, multi-variable, standard multi-variable calculus.

So this is partial of xi. So this will give me, I am just going to write this as xi dot minus, okay. Just like I take differentials, this is how we have defined v dot anyway, okay. Alright, so what do I, I just carefully expand things here, continue to write this as fx plus gx xi plus xi minus k0 x transpose and I know that this xi dot is just u, right. So this is u minus del k0 del x times fx plus gx xi, okay.

Everybody is convinced this is fine. Yeah, I have simply substituted for xi dot and x dot, okay. Great, great. Now we start playing our fun tricks, okay. As of now, what do I know from the previous page? I know that partial of v0 with respect to x multiplied by f plus gk0 gives me a negative definite term.

Yes? Okay, and I am going to use that. What am I going to do? I know that I have partial of v0 with respect to x, f plus g xi, I am going to write this as gk0. So what do I get here? Partial of v with respect to x fx plus gx k0 x minus, sorry, plus partial of v0 with respect to x gx xi minus k0 x. This is just the first term broken into these two pieces. Yeah, why? Because I know that this is something nice, right.

So we always want to rely on something that is already nice, right. And then I have of course psi minus k0 x transpose u minus partial k0 with respect to x fx plus gx psi. Yeah? Alright, great. So this I am going to use the previous page to say that this is less than equal to minus wx. Yeah, that is the first thing.

And then you can see that this term also has sin minus k0 x, right. So I can combine it with this term, right. How? This is just a scalar. Notice this v0 was a scalar. So every term in v0 dot is a scalar, right, just a scalar.

So transpose of the scalar is the same scalar. So we use these things regularly, by the way. Remember this, write it down in your notebooks. Transpose of a scalar is a scalar. I know this sounds ridiculous, but you will forget it.

You will think why these can be combined. Yeah? So every time I get a term with psi minus k0 in the end and the same term in the beginning with the transpose, all I have to do is take a transpose, right. So end and beginning, same thing, right. So I am going to, and we use these tricks very very frequently, okay. So I can combine these terms.

u minus partial k0 with respect to x fx plus gx psi and plus g transpose x del v0 del x transpose. Yeah? Because I have just taken the transpose and that it all shows up inside, okay. So now in order to claim, so we already have, so this is, let me write it again, v dot is less than equal to this quantity, right. So v dot is essentially what you want to prove negative definite, right, in some sense. Now in order to claim that this is a control Lyapunov function, what do we need? Can anybody tell me? What do I need to now show in the right hand side? So v dot is obviously as you know that this is Lf v, I will say Lf bar v.

Yeah? So why I put the bars is because the drift is, you know, so in this case what would be f bar? f bar would be fx plus gxi and 0 and g bar would be 0 and identity. Yeah? Because this is the drift vector field here. Basically terms multiplying the control and terms not multiplying the control, as simple as that. Yeah? So f bar is this guy and g bar is this guy. Yeah? Control only in the second, very second state, so identity here and drift only in the first state, so 0 here.

Okay? So that is why I say this is Lf bar v plus Lg bar vu. So what do I need? What do I need now? If you look at this expression what do I need to claim that this v is a clf? Not g bar and f bar. Can you say that again carefully? Not g bar and f bar. No, no, no, no, nothing to do with g and f, g bar and f bar or g and f at all right? I mean in the sense there is something more.

Yeah? Yeah, go ahead. Lg bar v equal to 0 means Lf bar v is less than 0. Yeah? Not just g bar and f bar. They have no role as such. Okay. Okay? So this is what I need to claim.

So basically what, so the right hand side is exactly the same thing by the way. Yeah? Let's not get too confused. Right hand side is exactly the same thing. Everything multiplying the control is the Lg bar v. Okay and everything not multiplying the control is Lf bar v.

Yeah? Obviously these two are scalars in this case. Yeah? Because Lf bar v will always be a scalar because v is a scalar and Lg bar v will be a scalar in this case because there is only one control. Okay? So what is Lg bar v in this case? From the right hand side can you read out and tell me? What is Lg? Absolutely. Thank you very much. So this guy is actually equal to Lg bar v.

Okay? So Lg bar v equal to 0 implies what? So Lg bar v equal to 0 implies psi is equal to K0 x. Yeah? Exactly the thing that we wanted anyway. Remember? Yeah? Somehow it came back. Okay? And this if psi is equal to K0 x you know that everything here goes to 0. By the way these were all drift terms also.

Right? Why would we believe that? This multiplied by this was also a drift term. Was part of Lf bar v. Right? But because psi minus K0 x goes to 0 or is equal to 0 if Lg bar v is 0 all of these go away. Yeah? So what am I left with? This implies that Lf bar v is less than equal to minus Wx. Right? Which is negative definite by assumption.

Okay? Negative definite by assumption. Okay? I hope this is clear. Yeah? This is how we test for CLF. Okay? All we see is that if the term multiplying the control is 0 then what happens to the terms that are not multiplying the control? Okay? So in this case the term multiplying the control goes to 0 means this term goes away. Which means these terms also go away.

The only thing that's left is this guy. Alright? And that is negative definite by assumption. W was positive definite minus W is negative definite. Right? And this is enough to claim that Vx, Vxin is a CLF.

Okay? Alright? Excellent. Okay? Everybody is clear? Yeah? How we just constructed a CLF for a integrator system starting from a single system. Alright? Of course the questions on how do you get V0 and W and K0 all these remain. Yeah? We will look at examples of course. But it's a constructive way. Yeah? All you did was kept constructed an error and you added the error square, norm of error square.

Okay? Now if I was to ask you what would be a choice of stabilizing controller for the system looking at this guy? What would you say? What would be a good stabilizing controller if I want to stabilize this system? Xi system? What do you think? Can you use this expression? This is what is called Lyapunov reshaping. Right? You take a Lyapunov function, CLF is also a Lyapunov function. So you take this V, take the derivative and try to make the derivative negative definite. Right now, yeah, you said it's CLF and all that stuff.

Excellent. And you can use the Sonntag universal formula obviously. That's obviously one choice. But suppose I ask you just look at this and tell me what is the stabilizing controller. Can you? Why? What? E to the power t. No, I want you to give me an expression for control, U.

What will you do? Yeah, what will you choose as U if I want to make V dot negative definite? Anything else? So this expression you will make 0. So this entire thing gone. Okay, great. Does that make V dot negative definite? We are back there. I ask again, is V dot negative definite? Look very carefully.

What is V a function of? Okay, it's only semi-definite. Why? There is no psi. All states don't appear, not negative definite.

So not enough to make this 0. Something more. What else? What will you do? Great, making 0 is a good idea. We need something more. So first step is cancel this.

Good. Add something more to the control. What else? You can get motivation from the expression of V itself, right? V is positive definite. I hope you believe that. Right? I mean, okay, we never, by the way, we never discussed this very carefully. So maybe I should go back there. Do you believe that V is positive definite? Why? V naught x is positive, right? Great.

But if you remember, we discussed this that if you have sum of two states and square it, if it's like x1 plus x2 square, it's a problem, right? Because for x2 equal to minus x1, it goes to 0. So how do you claim? If I ask you to claim a bit carefully, not just because it's a norm square, how will I claim that this is positive definite in xn psi? Remember, go ahead. Correct. So V naught is definitely no problem.

No, no, no, no, never say that. That is the same argument as saying that if this is not there, it's positive definite. Is it positive definite if this is missing? No. All variables have to be there.

Okay. So sure, in x it's positive definite. I mean, there is nice positivity in x. Great. What about the xi term? You have to use the same test.

The test is still the same. Okay. That it has to be positive everywhere, but at 0. At 0, it has to be 0, which it is.

V0, 0 is 0. K0, 0 is 0. So xi is 0. So xi is 0. So no problem. At 0, it is 0. Okay. If x and xi are non-zero in any combination, then this has to be strictly positive.

Is that true? How do you prove it? It is true, of course. It makes sense to construct this. How would you claim it? Both are positive, obviously. So nothing can cancel each other.

Right. For it to be 0, both terms have to be individually 0. So both terms have to be individually 0. So the first term being 0 implies what? x is 0. The only way first term is 0 is if x is 0. If x is 0, K0 x is 0.

So if the second term is 0 implies xi also has to be 0. That is the only way. This is how you will justify. Okay. Whenever we ask about positive definiteness, you have to be very careful in this argument.

Okay. First of all, and again, you look at how easily you get back to the old habit of saying, oh, it is because it's positive, it's positive definite. Okay. As soon as I asked you, you said this is positive definite. It's not. So it's very easy to slip to old habits where you say that even if not all variables appear, it is positive definite.

It is not. Okay. So that's the first thing. Okay. All variables have to appear for a function to be definite. Otherwise it's not definite.

It's only semi-definite. Yeah. Keep this in mind. Second, you have to do the usual test that for all non-zero states, it has to be greater than 0. Okay. So the only way V can be 0, because each of these is a non-negative term, is that each term is 0. So if the first term is 0, you know that x has to be 0, only choice, because V0 is assumed to be positive definite.

Right? Now if x is 0, this guy goes away. So this is just now half xi square. So for this to be 0, xi has to be 0. Therefore, the only way the second term is 0 and the first term is 0 is only if both x and xi are 0.

Okay. So this is a very clean evidence of positive definite. Excellent. Can you use the fact that this is positive definite to motivate how to choose the control here? What do you think? He's already suggested that you know, you cancel these two terms, which is good, because I don't know anything about definiteness of these terms. So it's the smart thing to do. You can cancel these guys. Say that again. But this is already cancelled, right? I've already removed these using some parts of U.

I mean, I'm going to basically I'm going to make U as plus del K0 del x fx plus gxi minus g transpose del V to 0 del x transpose. So these two cancel out, but I can add more terms in U. What more should I add in U? Absolutely.

You just introduce a minus xi minus K0 x transpose. Okay. If you just introduce a xi minus K0 x transpose, what will I get? Minus norm xi minus K0 x whole squared. Okay. It's just now the combination becomes negative definite.

Okay. The combination is now negative definite. So what is using just this is and this is the standard Lyapunov reshaping. I'm going to write it here now. Control law by Lyapunov reshaping is what? Is U equal to, so I basically cancel minus del K0 del x, sorry, plus del K0 del x fx plus g x xi, which is to cancel the first term.

Then I cancel the second term. This cancels the second term. And then I introduce. Okay. You can verify the dimension. Dimension will turn out well. And this will give me V dot as less than equal to minus w x minus xi minus K0 x whole square, which I know is negative definite.

Right? Because the first term, this is already definite. So both terms have to go to zero individually, which means that x goes to zero and xi goes to zero just by the same argument as before. Okay. All right. So this is a valid control law.

In fact, this is how we design control laws most of the time. Yeah. Most of the times this is how we will design control laws by Lyapunov reshaping. We don't usually go back to the Sontag universal formula, mostly because the computations are very complicated. Yeah. Now if I put all the square roots and stuff here with this expression, right? You see what is the, notice in this case, this was Lg bar V.

Great. Nice. But what was Lf bar V? Lf bar V is this guy, this guy, this, this and this put together.

Okay. This was Lf bar V. Okay. Very, very painful looking. So if I wanted to use the Sontag universal formula, of course, it's a very painful calculation. Of course, it's not, I mean, if you implement it numerically, this is not a big deal. Yeah. So you will just compute these as Ax and Dx and then you will just at every instant in time you compute Ax in one place, Dx in another place and then you just compute this whatever, this minus the universal formula.

Okay. That's pretty straightforward. But if I wanted to actually write it and show things with it, it becomes very difficult. That's all. Okay. And in fact, in this case, I know very much that this is also, because of this construction, I actually use Lyapunov reshaping.

I know that this will turn out to be a smooth controller also. Okay. Whereas the universal controller will only be almost smooth. Yeah. Here I can guarantee that it's a smooth controller just by looking at these expressions. Alright. So, most of the time we use this kind of a Lyapunov reshaping. Alright. Thank you.