Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 6 : Lecture 33 : Control Lyapunov functions-Part 5

So, what do we do? We want to apply the implicit function theorem. What do we do? We are only looking at the scalar case here by the way. We have assumed that A and B are scalars for this proof. You can also do the vector proof but I am just keeping the discussion simpler by assuming scalar A and scalar B, ok, which means what? That the essentially means that your states are scalar states, ok, alright. So, what do I do? I basically write this as a function. This is the idea.

This is that function, ok. You will see why. If you look at this function, it is a function of A, B and Z. You already know the AB.

I have used the same notation where it was AXBX. Here it is AB. I have for the moment assumed that A and B are independent variables themselves and the X dependents are forgotten. It is actually irrelevant. So, we do not write the X dependents of anything because that way A is dependent on X, B is dependent on X and Z is also dependent on X.

But the fact is when I write this expression, X does not explicitly appear anywhere. This is just a function of A and B, right. So, I leverage that to just forget the X dependents and I write a function H, ok, and this function is BZ square minus 2AZ minus B cube equal to 0. Why did I write this? Because the solution of that is this guy. If I solve for Z, you see this is a quadratic in Z, right, quadratic function in Z.

If I equate it to 0 and I solve for Z, this is exactly the expression I will get. This is the control except for the negative sign. Again negative sign is irrelevant. Positive is smooth and negative is smooth. So, this is actually the expression for the control in the scalar case, right.

So, that is why this H has been chosen, not chosen by some magic or anything, pretty straightforward. Alright. So, what do we say? We consider this function H of A, B and Z, three variables. It does not matter. Like I said, this X and Y were different dimensions.

So, I can club these two as one variable and this as one variable. Pretty simple, does not matter, ok. This is scalar function of three variables equated to 0. Now, I also consider the set S in R2 which is a set of all AB except this. Why do you think I removed this point, this particular region if you know? Why do you think I removed this from set S or from R2? This is the case that cannot occur by the CLF definition, cannot occur by the CLF definition, right.

Because if B is 0, A has to be negative or whatever, yeah, that is the CLF requirement, yeah. So, although I have forgotten that they have a function of X, but I have to remember that there is a CLF requirement. The CLF requirement says that if B is 0, A has to be negative. So, this is not a case that can appear. Therefore, I ignore this region.

What is that region? So, essentially it is this entire blue thing that I have drawn except for this axis, the right half. So, this is the right half, this entire line is not there. Everything else is there in the set S, okay, clear? Okay, great, great. So, now I have essentially solved this, okay, solved this equation. And now I can write this as tuples, what is three tuple, yeah, AB and Z as a function of AB, okay.

This is the, from the implicit to the explicit. This is the implicit and this is the explicit version, okay. Here what? ZAB is defined as 0 when B is 0 and ZAB is defined as this guy when B is non-zero, okay. Exactly the universal formula for the scalar case, okay, exactly the universal formula for the scalar case. Now what? Pretty simple.

I compute the, here I needed to do del F del Y. In this case, I need to do the, take the Jacobian with respect to Z. This is the Jacobian. I hope you are used to this. It is basically taking partial with respect to multiple variables when you take and it is a Jacobian.

Here I am still taking with respect to one variable but still I would use the word Jacobian, okay, because you will use it in several places in the future. So what is it? I take partial with respect to this Y variable. In this case, the Y variable is actually the Z in this case, yeah, because I am writing Y as GX, right. So therefore this is partial with respect to Z that I have to verify. What is that? It is just twice BZ minus twice A, yeah, okay.

And now I compute it for these two cases, okay. So twice BZ minus twice A, so when B is equal to zero is just twice, minus twice A, right. If B is equal to zero, this guy is gone, okay. But when B is non-zero, I have to substitute Z here in terms of AB because these are my independent variables if you will. I am resolving in terms of A and B.

So when B is non-zero, I substitute Z here and I get this guy, okay, and I get this guy, okay. The good thing to see is I have mentioned that this is non-zero for all AB in S but actually it is positive, okay. It is actually positive for all AB in S because this is positive, this is also positive because I have chosen the positive one, I mean that is just notation here, okay. So but the important thing is non-zero which is what we require. Full rank, in this case full rank is just non-zero, scalar, the scalar, right.

The Jacobian is a scalar. So full rank is non-zero and it is. So I can immediately invoke the Implicit Functions Theorem to say that Z AB is the unique solution, further Z AB is smooth. The only thing we need to note is that when definition 2.7 holds, your A and B are always in the set S.

That is if you have a CLF, then your AB are always in the set S, okay. So therefore on this set S, I have already shown that Z AB is a unique smooth solution, okay. On the set S, this Z AB that we have is a unique smooth solution. So this is a very, by the way in general mathematics analysis and all, this is a very standard way of proving smoothness, okay. This is how we do it using Implicit Function Theorem.

Somebody asked me to prove smoothness of some solution or something. Just think at the back of your mind think that Implicit Function Theorem has to be invoked, okay. So then the only big creativity here is constructing this sort of a function. That is the creativity here. That I constructed this function because the solution of this function gives me the universal control formula in this case.

In a different context, it might give me something else. So the only creativity required is to construct this smartly. Once you do that, all you have to prove is that the solution is what you are looking to prove to be smooth and the Jacobian is or the derivative or the first partial is non-zero or full rank. That is it, okay.

And we have done that, okay. Alright. Anyway, continuity at the origin is anyway consequence of the small control property. So we are not, you know, concerned about it. Remember that this analysis is not for the origin itself because these conditions are stated as if X is not equal to 0. All the CLF definitions are essentially saying if X is not equal to 0, this happens.

X is not equal to 0. So they are not for the origin. So the continuity at origin is basically from the small control property, okay. Nothing else.

That is the idea. Alright. Excellent. Let us try some. I am also going to try with you. How we can construct some CLFs and we try to get some controls out of it and so on and so forth, okay.

Alright. So we start with simple things. This system, okay. We have been doing this, working with, I mean I have shown you different forms of the system without the control. Now I am giving you something with the control.

It is a double integrator. It is a double integrator dynamics. Very relevant because a lot of mechanical systems can be reduced to double integrators, okay. Alright. So position and velocity states, derivative of second state is acceleration and typically acceleration is what you control, okay. So very simple connection with mechanical systems.

What do you think will be, what do you think I should choose as my V, as my control Lyapunov function? So do you remember what we chose as half x1 square plus x2 square? Let us revise for ourselves the CLF definition so that we do not. What do we want with the CLF? It has to be first of all this, right. That it is a candidate Lyapunov function. This is too easy.

Most functions we choose are CLFs. The next one is if the contribution of the control terms are 0, then we want the drift vector term to be strictly negative for all non-zero states. All non-zero states this has to, at zero state this can be zero, no problem. Nobody cares because you are already at the equilibrium. But when the state is non-zero, this has to give a negative contribution. This is what is the requirement for a CLF, okay.

So let us keep that in mind when we try to design. Okay, fine. Let us try this simple one. What is it? Let us try half x1 square plus x2 square, okay.

Half x1 square plus x2 square. So what do we do? We try to find the, first of all what is the drift vector field here? What is the drift vector field? We will do all the competitions formally. There is a quick way also but I will not do the quick way. See the quick, I will tell you what is the quick way. Quick way is compute V dot. Then you have x1, x1 dot the way we were doing earlier.

X2, x2 dot. This is x1, x2 plus x2 times U. So whatever multiplies V dot is A. This is Bx. This is the simple quick way. If you want to do the longer way, you have to write F0 x, F1 x, del V del x and all that, okay.

But this is the quick way. Because if you see this is how it is comes out to be, right, every time. You see this expression right here, this guy. This expression is illustrative enough. So this V dot is always Ax plus B transpose U.

So using this I can always compute A and B, okay. And what do I want for a CLF? I want that. What is the CLF requirement now? Well, first of all it is a candidate Lyapunov function anyway. So I mean that is oops, oops, oops, that is done. I want what? What is my requirement for CLF? Second condition in terms of A and B, in terms of A and B because I have written already A and B, right. For all x not equal to 0, I need what? Yeah, if B of x is actually 0, then A of x has to be negative, okay.

B of x is actually 0, then A of x has to be strictly negative, okay. Let us investigate. This requires investigation. B of x 0 implies what in our case? It implies x 2 is 0, okay. That the only way that B of x is exactly 0 and this implies what? That x 1, x 2 is also 0 which is equal to A of x, right.

And this is a problem, right. It is not negative, right. Not negative because it is actually 0. A of x turns out to be actually 0. So not a CLF, not a CLF, disappointing.

Not a CLF. Such a simple example we started with. We failed here. So annoying. Do you, huh? See, you understand that what do we need? We need that whenever Bx is 0, the A has to be negative, strictly negative for all non-zero x. So when is B 0? When x 2 is 0.

Only way for B to be 0 is x 2 0. It is in fact if and only if, right. There is no other way about it. Now I want A to be negative but let me see what happens to A.

A is x 1 times x 2. So whatever is x 1, it is irrelevant. Even for non-zero x 1, this is 0 which means A is exactly 0 but I want A to be negative, okay. So this is a problem. So this function is not a CLF. Now what? Do any of you remember all the modifications in VI did for that when we analyze this kind of a system? We analyze this kind of a system. What is a good control by the way for this system? Anybody remembers? We did this, no? I mean I did not write the U, I did not use the letter U but I gave some system, a double integrator system, linear double integrator.

What was it in the second dimension? What is the control you would give for this system to stabilize you think? Minus x 1 minus x 2. This is the damper, spring mass damper. Go back to spring mass damper, okay. Like remember a target system and try to match it with the target system.

Target system is spring mass damper. So I just give you as minus x 1 minus x 2. That seemed to work. Do you remember what kind of function I used to prove that? That the system was stable in that case? It was complicated, not very straightforward. It was not straightforward. If you remember x 1 square plus x 2 square did not work because it is not a strictly Lyapunov function, okay.

I do this. I do this. Anyway I will see what happens. I am not sure what happens but yeah. So I compute V dot again. I will try the simpler way of doing thing. So this x 2 plus x 1 times x 2 dot plus x 1 dot which is U plus x 2, right, plus x 1 x 1 dot which is x 1 x 2, right, okay.

I just computed the derivative and substituted things. Now I know that everything multiplying U is the B and everything not multiplying U is the A, right. So what is the Ax in this case? Ax is x 1 x 2, twice x 1 x 2 in fact plus x 2 square, okay. And Bx is, right, it is x 1 plus x 2, okay. We do the same analysis again. For all x non-zero what happens? If same thing I rewrite Bx is 0 we need Ax to be negative, right.

Now when is Bx in this case 0? x 1 equals minus x 2, correct? Correct? Okay. Now this implies what? 2 x 1 x 2 plus x 2 square is actually equal to what if I substitute x 2 equals minus x 1 or x 1 is minus x 2 this minus x 2 square whichever one, write either one. This is equal to Ax and this is negative because x is non-zero, right, negative, okay.

So this is good, yeah. This is a CLF. If you look back at your notes and you check out what we did for this spring mass damper example x 1 dot is x 2, x 2 dot is minus k 1 x 1 minus k 2 x 2. Just k 1 k 2 scaling was there. This is exactly this, right. We had this kind of a function, okay.

So you know that this is not a strict Lyapunov function, okay. So this is not a CLF either, okay. CLF is something more. A CLF lets you choose a control, yeah. In this case you would not have been able to choose a control, okay, which will show you stabilization with the Lyapunov function. Of course with LaSalle invariance, yes, but the entire idea of CLF is based on Lyapunov theorems and Lyapunov functions, not on LaSalle invariance.

LaSalle invariance as you remember is a more very general sort of a result. It does not align with how the Lyapunov theorems go. Lyapunov theorem you can just recall how the proof of Lyapunov theorems went, how the proof of LaSalle invariance went. Zero connection between the two, yeah. Zero connection, like completely world apart. In fact for LaSalle invariance you do not even need to start with the positive definite V.

That is not a requirement. Positive definite V is not a requirement, yeah. So these rely on Lyapunov theorems. Therefore, wherever the Lyapunov theorem fails, it will not turn out to be a CLF. It will be something different, okay. It will be a, it has to be a CLF, yeah.

So in this case this is what is a CLF, okay. So I think I, that is why I have given you, okay. I gave you a nice hint for this system. This is a small extension of what we did, okay. So this is how you, this is the procedure though or I mean of course choosing this is still a little bit of a guesswork.

But this is a lot of procedure how you check, okay. Now, absolutely, absolutely. It is not like there is no easy path there. If I give you a problem, I will give you a hint. Also can you guys just tell me or try to guess without going into what I gave you as another V. Forget what I gave you as another V. Do you, do you folks think you can guess what I can add here? Suppose, let's see, notice that this pieces are inside this V, right.

X2 square plus X1 square is also here, right. There are some additional terms. That's all. So forget what I gave. Can you add some term here to make it a CLF to this guy? Mixed terms are allowed. So, so remember, how are we, so this is one nice hint I will give you that we are typically choosing Lyapunov functions as quadratic forms, right.

I mean, we remember we made this equivalence. If you have a positive definite matrix, then you have a positive definite function. So therefore Lyapunov functions can be those quadratic forms. So your V is, can be typically in this form X transpose PX. Now we have exploited a very small, now what do we need? We need that, so what do we need? We need that this P be positive definite, correct.

We need this P to be positive definite, alright. Now this is a very, very simple example of this, right. It is just one in the diagonals and zeros here. Too simple. I can have more complicated version where I have something in the diagonals, but still this is positive definite, right.

All I have to do is check the determinant, right, for this 2 by 2 case. I just have to check the determinant. Pretty easy, right. So for the two dimensional case, all I need is this is positive, this is positive and the determinant is positive.

That's it. So mixed terms are allowed. All I am trying to say is mixed terms are allowed as long as the magnitude of the mixed term is small, ok. Now do you think that will help us? Not an x2 square term.

See the only thing I can add here is the mixed term. I can tell you that. There is no other. Yeah, because x2 square is anyway already there. So what, ha. So now suppose I add this alpha x1 x2 because this is the only other unique term I have.

Of course you can do funny thing like sine x1 sine x2 and all that. That's up to you. I wouldn't go there. I would start with these kind of things, exponential x1 x2.

One can probably think. Maybe it will work also. I am not denying it. Yeah. But I am saying you should start with this simple case and then try to add terms to it rather than try to guess this guy. This is coming from some fundamental logic which I know, you don't know as of now, ok. You will know soon but right now you don't know why this works, ok.

So for you it is simple thing to add some alpha x1 x2. Suppose I did. Now what happens? I add some terms in the derivative, right. So I am going to delete this just to make some space for me. So this is plus what alpha x1 dot x2 plus alpha x1 x2 dot.

Let's see if this works. And then what? Alpha x1 dot x2 is alpha x2 square and this is alpha x1 u. Already nice things have happened. You see why? Now if Bx, now Bx is not just this, Bx also has this, ok. So Bx equal to 0 implies x2 is minus alpha x1, right. x2 is minus alpha x1, ok. Now what do I get for Ax in this case implies Ax is what? It is minus alpha x1 square from here from the first term.

From this guy I will get alpha cubed plus alpha cubed x1 square, right. So if alpha is less than 1 I am done, alright. Yeah. So what is this? This is basically minus alpha minus alpha cubed x1 square. So if alpha is less than 1 I am good, right.

And I can also, I believe if alpha is less than half, this will remain positive definite. Less than half? Whatever that is, I mean greater than 0 for all alpha what? One fourth minus alpha square, 1 less than half. This will work. I am just computing the determinant.

It has half half and alpha by 2 alpha by 2 actually. So it is one fourth minus alpha square by 4. So alpha less than 1.

Did you just say that only? Oh thanks. Oops. You were right.

Alpha less than 1 works here also. And here also alpha less than 1. I apologize. Sorry. Ok. Alright. Yeah. Yeah. So alpha less than 1 works, ok. So all you need to do is alpha less than 1.

All you need is add this one term whereas I added two terms. Whatever I made slightly more complicated things. Yeah. So do not go by what I constructed. You just add terms. Start with the usual quadratic, simplest quadratic x1 square, x2 square, x3 square.

Then you add terms. You will get something. Ok. It is not that bad. And once you constructed a CLF, ok, you can construct a controller. Universal formula. If nothing else works, universal formula. If you cannot guess, universal formula.