Nonlinear Control Design

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Week 6 : Lecture 31 : Control Lyapunov functions-Part 3

So, let me sort of recap very quickly what we have talked about in the context of CLFs, control Lyapunov functions. So we very conveniently and comfortably use the notation CLF or the terminology CLF. So please don't get confused, it just means control Lyapunov functions. We started with a general non-linear system like this and this was the definition for the control Lyapunov function. The first one, the first condition basically required that it be a Lyapunov candidate, a valid Lyapunov candidate and the second condition is sort of the nice negativity condition. So you in the absence of the control you had the negative definiteness condition.

Here what all we are saying is that we are saying you can actually find some control U such that you can make the derivative negative for every state. That is really what is a CLF. It says that this function is such that it allows you to find a control such that for that value of control the V dot is negative and this you have to do point wise. This is a point wise computation because so the x here is fixed.

So you notice that because you are taking infimum only over the control, the x is actually a fixed quantity. So once you fix the state, it is just some number that you have. Basically you just have a function of U in here and then you are just trying to find an infimum. In fact you can just think of this as an optimization problem. It is actually almost an unconstrained optimization problem.

I mean the only thing is you want the value to be negative. You want the value to come out to be negative. If it so happens that you run an optimization on this and the value turns out to be positive of the function here of the cost if you may have a V dot then it is not a good V at all. So the V is not a CLF. And of course such V such a CLF is very useful in control design.

We will look at it because you can see that it allows you to design a control which makes the V dot negative. And V dot being negative definite is what you require for asymptotic stability. So obviously that is what you want. Now we then specialize this to the control affine case. Why did we do that? Because we realized that, well we did not realize anything but the researchers who have been working on this for several years, they realized that with just a general non-linear system where the control can appear in any form, not necessarily in an affine form, then it may not be possible for you to design continuous controllers.

Virtually impossible. So they figured that it is not possible. So then they specialized to systems that are linear in the control. And hence called control affine systems. So this is the structure.

And this structure is the most universal structure in non-linear control. Here the F0 is the drift vector field. Fi's are control vector fields. These are basically state dependent vectors. At each value of the state, they give you a direction, a velocity direction.

And then you have a control scaling. So it is almost like saying that I have, if you count these, I have F0 that is 0 and then I have 1 to m. So I have m plus 1 such vectors. I have potentially m plus 1 velocities. Notice that m is necessarily less than n.

So the state space is n dimensional and your number of velocity vectors that you have to play with is m dimensional. So m is less than n. So m is typically less than n. Which means you have less velocities than the number of dimensions if you think of the state space as a dimension. It may be less than the number of dimensions.

So the idea is can I play around with these velocity vectors so I can go in the right direction. So if you just think about moving on a sphere for example, the surface of a sphere for example. Now suppose I want to move on the surface of the sphere. This is my requirement. And I have say vectors in all three directions.

Velocity vectors in all three directions. So I can potentially if I specify this vector in a bad way, I can potentially get thrown out of the sphere also. Instantaneously I could just be thrown out of the sphere which is not okay. So the idea is can I play with these vectors so it would be something if I want to draw some picture like this. Suppose this is the surface of the sphere.

I will have a vector this way. I may have a velocity vector this way. Alright. I may have these three velocity vectors. Now as long as my actual velocity is in this plane, I am more or less okay.

Because this plane is the tangential plane to the circle or to the sphere. I am more or less okay. I will remain on the sphere at least on the surface of the sphere. But if I start doing anything in this direction, I get thrown out of the sphere.

Okay. So what would my control try to do? My control is just a scaling. All it is doing is it is scaling each of these fields, each of these vector velocity directions. Right. So you are just in fact you are doing a linear combination of these velocity directions. To get the direction you want to go in.

Okay. So although we never design controllers in this thinking like this honestly speaking, nobody designs control like this. Very difficult to do. But this is the logic by which the controllability of a system is defined. Okay. If you cannot reach all possible directions, then you will have some issue with the controllability.

Okay. That is the idea. Alright. So what we did was we specialized to control affine systems and for that we defined the equivalent version of the control Lyapunov function. Okay. We have already proved equivalence. Well at least we proved one side, the other side was supposed to be our homework which I will assign soon enough.

So this is what is it, what does it say? The first one is again exactly the same thing as before. The second definition changes a little bit. Okay. Nothing significant. It just says that if the contributions of the control vector fields are zero, then the drift vector field has to push you in the negative direction.

That is it should make V dot negative. Okay. If not, then again in a sense what we are trying to say is the system is not stabilizable at all. Okay. You cannot make the system go in a good way.

Okay. Behave well. Okay. So that is the whole idea. So we proved again one side of the equivalence. Then further we talked about the small control property. This was the final sort of property that is required to design continuous control laws.

Okay. What is the small control property? It just formalizes and we saw it with a very nice example. Right. That for a system like this the control becomes larger and larger as you come closer to zero. Okay. And zero is a equilibrium of this system.

Okay. So which is very bad. Right. Because if you want to try to reach the equilibrium from both sides, you are going to give larger and larger control efforts which is sort of ridiculous. You don't want to do that. So this creates a discontinuity at the origin. And in order to prevent this, you say or you assume that the system has a small control property. And what is the small control property? It basically says that if you start with small values of the state, that is norm x is less than delta, then with small values of control, that is control is close to the equilibrium control, you can make this V dot negative.

Okay. So basically it says, it essentially says what we don't have here. Yeah. Essentially says what we don't have here. That if you are close to the equilibrium, then the small values of control should sort of send you towards the equilibrium.

Okay. That's the whole idea here. And this is the small control property. We already sort of claimed that this small control property is stronger than the second condition of CLF. Yeah.

This is a stronger requirement. Yeah. Why? Because if this holds, then you know that if all these terms are zero, all the control terms turn out to be zero, then this term is still negative. Okay. So this is essentially, so this implies the previous definition is satisfied.

Okay. So small control property is a stronger requirement than the CLF property.

Okay. All right. So once we have this small control property, this is where we were last time. Archtyne and Sontag, it is basically their work, primarily work by Archtyne and Sontag. Yeah. They were the ones who started talking about the CLF, the small control property and then corresponding control to them.

They gave a universal controller. Okay. One of the coolest things about this result is that unlike a lot of other mathematical results and this result I tell you is very mathematical. They actually give a constructive design of the control. Okay. We already saw this last time. So anyway, what it says is if you have a control affine system with the control Lyapunov function as per this definition, then if the system admits, the system admits a small control property if and only if it admits almost C infinity stabilizer.

Okay. And we clearly said what is almost C infinity. It means that smooth everywhere in a perforated neighborhood of the origin and continuous at the origin. Okay. It means that the control that you obtain will be smooth everywhere, infinitely differentiable everywhere, but at the origin it is only continuous.

Okay. Not smooth. Not smooth at the origin. Okay. It is continuous at the origin. So this is what you can achieve. And remember in specific cases like example that we will look at or we can look at, you will find smooth controls which are smooth at the origin also. Okay. But remember this is a very, you know, a result which covers all such control affine systems.

So it is a very general result. Okay. Therefore, they are saying in general you cannot claim this. That you will always find a controller which is also smooth at the origin. Okay. So what you can claim is it is almost C infinity.

Okay. But in the examples that you will see, you might find smooth controls. Okay. Using this formula itself. So it is not, you know, not that this covers all cases.

Okay. So the Einstein-Sontag, so this result, the proof of this result is based on the Einstein-Sontag universal formula. Okay. Or the universal controller or the universal formula, whatever you wish to call it. Yeah. It is defined by first defining these two placeholders, Ax and Bx.

What is Ax? Ax is the derivative with respect to the drift vector field. And Bx is the derivative of V with respect to the control vector fields. Yeah. So it is tagged as a vector.

We already saw what are the dimensions. What is the dimension of this guy? We discussed this right last time.

R1. It is a real value. Okay. And Bx is? This is Rn. It is an m vector.

Okay. All right. Great. So what is the universal formula? This. This is the control. Slightly complicated looking. But this is the control. What is it? Ux is minus negative of Ax plus square root of A square plus norm of B4 Bx divided by Bx square if Bx is non-zero.

And if Bx is zero, then the control itself is put as zero. Okay. So you can see that B is a vector. So therefore we are being careful. Whenever we take a square of B, it is the fourth power of the norm of B.

We are taking the norm of the vector. Okay. And here also we are dividing by norm square. So this is B.

Okay. So you see that this whole thing is in the direction of B. Okay. This whole thing is in the direction of B. Okay. Because B is actually a vector of dimension Rn. Notice, this is correct.

Why? Because control is also required to be of dimension Rn. Okay. Control itself is Rn. We have m controls.

So we have m control vector fields. Okay. So this dimension is okay. Right. Because B is also dimension Rn.

We just discussed this. All right. So dimension wise, no problem. Yeah. What is the significance of Bx being zero and Bx non-being zero? This is the clf condition. Right. Because Bx was what? This guy. Bx is del VA, del V del x, Fi x for all i stacked column vector.

And if this is zero, it means that del V del x Fi x is zero for all i. Yeah. I should write this. Yeah. For all i from 1 to m.

Okay. So when is B zero? If del V del x Fi x is zero for all i. Okay. So this is the control Lyapunov condition. Under such circumstances, you know that the drift vector field itself will give a negative V dot. Okay. So we put no control. Because anyway if the contribution of the controls is zero, because here if you put any non-zero value of control, it is useless.

Because the drift, the V dot is going to be zero even if you put non-zero values of control.

Right. Because you have Ui Fi. Okay. So it is irrelevant. So we put the control as zero vector itself. And here we give it some particular value. Okay. In order to verify that this is in fact a stabilizing controller, because that's what we want, we can just take V as our control Lyapunov function itself and compute a V dot.

Okay. So V is our candidate Lyapunov function. We already know that the control Lyapunov function is a candidate Lyapunov function. So I take that as my V for the system for doing Lyapunov analysis.

Right. And then I take a V dot. What is V dot? Is exactly this. Partial of V with respect to X. And this for this control affine system.

Right. And then you know that this is actually AX. Right. And this is actually B transpose U. Is that clear? Right.

Because B is this. And B is this guy. So B transpose U is exactly this multiplied by this. Yeah. Okay. So once I have that, all I'm going to do is substitute for the control from here.

So again I get two cases depending on whether B is zero or non-zero. Alright. So when B is non-zero, you can see what will happen.

I will have AX plus this guy. Sorry. I have the control here. Right. So I'll have my AX and then BX transpose of this.

So BX transpose times this. Okay. So this is a scalar. So BX transpose just moves here. Right. BX transpose just moves here. And BX transpose BX is what? Norm of BX whole squared. So this guy will, this guy cancel out.

So all I'm left with is AX minus AX plus this guy. So AX minus AX cancel out again. So I'm left with just this much. As expected because V is a scalar, so V dot is also a scalar.

Important thing to note is this is strictly negative. Yeah. Because BX is not zero. So this is strictly negative.

Whatever AX is irrelevant. Just because BX is not zero, this is strictly negative. Yeah. Alright. And if B is, BX is actually equal to zero, what happens to the control? Control is just zero. So I'm left with just AX.

Okay. But I already know by my assumption that AX has to be negative because V is a CLF. Right. V is a control Lyapunov function. So AX which is defined as this has to be negative when these terms don't contribute.

When BX is zero, A has to be negative. So when B is zero, this has to be negative. Okay. This is true only when B is zero by the way. When B is non-zero, A can be negative, positive, whatever.

Okay. But when B is zero, A has to be negative. And so what have I shown? That V dot is negative. Yeah. And in fact this is true for all non-zero X. Remember in entire CLF definition, although we didn't stress on it much, everywhere you see that for X not equal to zero, for X not equal to zero.

Okay. All these assumptions are for non-zero X. Okay. So it works out nicely. So V dot comes out to be negative definite. And this means by our Lyapunov theorem what? What does it mean? V dot turns out to be negative definite. Asymptotic stability. Done. Right. The system is asymptotically stable. Alright. Thank you.