Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 6 : Lecture 30 : Control Lyapunov functions-Part 2

So, we start focusing on control affine systems like you said it is affine because there is no control connected to this term. So, it is not even linear in the sense linear is defined. Alright. We assume because we well we are interested in looking at equilibrium the origin is the equilibrium we definitely assume that there exists a equilibrium control if you may. Yeah. Such that this happens.

Okay. So, we assume existence of a u bar such that f 0 0 plus summation of u i bar f i 0 is actually 0. Okay. So, we assume some equilibrium control existence also.

Okay. Why did we do all this? Because we could not get a nice smooth feedback that we were looking for. Okay. So, at least C 1 feedback that we were looking for. We only got a feedback that was C 1 and everywhere but at the origin.

Okay. At the origin we had you know we had this lack of differentiability. Alright. And it so turns out I mean you can try different examples you can try different controls here for this particular example it so turns out that for this kind of a system you will never get a feedback law u which is you know smooth at the origin C 1 at the origin. Okay.

So, in order to give conditions for nice smooth controllers also at the origin because origin is actually a point of interest we do want to go to the origin. You had we had to specialize to control affine systems. Okay. That's why we are looking at control affine systems. Alright.

We state an equivalent version of the control Lyapunov function definition for control affine systems. I say this is equivalent for this system and we will actually prove one half of it other half is actually an exercise. Yeah. How do we redefine control Lyapunov functions? The first one is still the same. It requires a function to be a candidate Lyapunov function.

Yeah. So, it is still the same. Exactly the same. Alright. Now, the second condition is where things change.

Yeah. For the control affine case. If the this statement basically says that if the contribution of the control terms is zero. Yeah. If the contribution of the control terms is zero to the right hand side then the drift term has to give negative V dot. This is what this is saying.

These terms are what are connected to the control. Right. When I take V dot I will get del V del x F0 plus Ui del V del x Fi. Basically this expression that you see here on the left.

Okay. So, del V del x Fi are the terms that give you the control movement due to the control. Now, if it so happens for some x that all these terms are zero then the control cannot move the system. Right. In that case we require that the drift itself moves the system in the direction of the equality.

Okay. So, when the control terms do nothing then I need this term to act and give me a negative quantity. Okay. So, that is what this is saying. Okay. How is this equivalent? I only prove one side like I said.

I will assume this and prove the previous one. Okay. The other way around is the exercise. So, if I assume this what am I saying? Let us choose an x bar such that this happens for some non-zero x bar of course.

x bar is non-zero. Okay. This is what is this assumption. Okay. And I have for this particular x bar I have that this happens.

Okay. But then things are very after that it is very easy. Right. Because if this guy is negative and this is zero this entire expression for V dot which we had in the first theorem. Right. In the first definition we had del V del x f x u.

Okay. And what is del V del x f x u in this case? It is precisely this guy. Yeah. Del V del x f zero plus summation u i f i. Right. So, if there is no contribution from this guy so this guy is zero.

But then by this assumption I already have del V del x f zero is negative. So therefore this guy is negative. Okay. So even though I am not actually taking any infimum over u, taking any infimum over u is pointless in this particular case.

Right. Because del V del x times f i is zero. Yeah. So, there is no effect of the control at all. But the drift term gives me a negative outcome which is what I need.

Okay. Now the other case where these terms are not zero. What happens if these are not zero? Okay. The only two possible cases are this. Some at least only two possible cases are this is zero for all i.

Okay. And the other possible case is for some i this is non-zero. For some i. At least for some i this is non-zero. That is the other possible case. There are only two possible cases.

So what happens if del V del x f i is non-zero for some i? It is pretty straight forward. See look at this again. This expression of the V dot for some x star. Okay. If I just think of one control just to illustrate that there is only one control, one control vector field then this expression looks so.

Alright. Now I already, so then if I expand it I have this guy. Okay. If I expand it I just have del V del x f zero plus del V del x f one times u one. But I have already assumed that this is not zero anymore. We have already covered the case when this is zero.

So now this is not zero. If this is not zero I can choose a control like this by inverting this guy. So I inverted this, cancelled this and inserted a negative quantity. Some negative quantity minus alpha. Yeah. So therefore if you compute V dot in the previous definition del V del x f x u turns out to be negative.

Okay. So I have actually given an expression for u. Yeah. Okay. Even if you had multiple controls and multiple control vector fields I know that for one i at least this is non-zero.

For one i this is non-zero. Then I will just make that u i to be this guy. Everything else I will keep it at zero. All the controls will be zero and I will just choose u i as this expression right here. Okay.

So then I have negative contribution. I am done. Okay. So this is how you can prove that this definition implies that definition for control affine systems. Okay. The other side that is the first definition implies this definition for control affine systems is what is the exercise that you have to prove.

Should not be too difficult. Now we have still not reached where we want to. Yeah. We still do not have this way of constructing nicer controls. Okay. We still do not have a way of constructing the nicer controls.

Okay. We still do not have the ways that are smooth at origin and things like that. Okay. We are still not there. So for that in fact we need something more.

Yeah. We are already at control affine systems. Yeah. We need what is called a small control property. So most of this work is due to Archstein and Sontag.

Yeah. And in fact the references are also here. So they actually you know proved all these results. You can see the years.

It is 83, 89. So not too recent actually. Yeah. So they sort of came up with this notion of small control property which is actually a strengthening of this control Lyapunov requirement if you may for a control affine systems. What does the small control property say? It basically says that if your state is close to the origin then your control also should be small. Okay. So it is a very reasonable requirement. It just says that your system should not be ridiculous that even though you are very close to your equilibrium you need very big controls to bring it back to the equilibrium.

Okay. So that is sort of what it says. So how does it again whatever we say in word we try to write in the math epsilon, delta kind of thing. That is what this is. Okay. It says for all epsilon positive there exists delta positive such that for nonzero x in the delta ball there exists some control vector which is epsilon close to the equilibrium control and V dot is negative.

Okay. And V dot is negative. I hope you see that this is stronger than the control Lyapunov condition. Okay. Why? Because in this inequality it should be evident that even if all of these are zero this is still required to be negative and that was the control Lyapunov condition. Right.

The second condition was the control Lyapunov condition. The first was just positive definiteness. So that is anyway you know there anyway. Okay. So the second the control Lyapunov condition this is stronger than the control Lyapunov condition.

Okay. This implies the control Lyapunov condition. So we need this condition to state any result on nice controls. Okay. And like I said this is a very obvious result but we still try to look at it with some nice very very interesting example. I mean these guys come up with very fun examples.

I can tell you that. They come with a counter example actually. If you look at this system x dot is x plus x squared times u. Okay. Now I hope it's sort of evident to you that if I try to construct a control forget v and so on and so forth. There is no I mean there is a v here sure but here we are not talking about the v.

Okay. Suppose I want to construct a stabilizing control here. All right. Close to the origin. You can see that first of all I will need a negative x. So basically this term has to contribute something like a negative 2x.

One possibility. If this term contributes a negative 2x then I have x dot is minus x and I know it's a stable it's going to go to the equilibrium. Right. Good to go. The only issue if I make this minus 2x and I try to compute a control out of it I may not be able to divide by x square all the time but the point is I will still have 1 by x type of a thing happening in the control.

Right. Okay. So if you look at it in a different way whatever control you have here is being scaled by x square. So when you move far from the origin or slightly far from the origin the control effect is significantly multiplied but as soon as you come to x less than 1. Okay. As soon as x becomes less than 1 you start getting closer to or norm x or absolute value of x becomes less than 1 you start coming closer and closer to the origin.

Yeah. The effect of the control is significantly shrunk. Yeah. Significantly shrunk. Okay. So even if you try to apply a minus 2x out of anything that I mean even tries to cancel this minus x this x with a minus x you will still have something like a 1 by x happening that's why the x square very specific purpose. So what does it mean? It means that you will keep having to scale up your control as you get close to the origin.

Right. As you get closer to the origin control will have to be scaled up further and further. Okay. I hope you are convinced.

So u is large for small x first thing. Second thing. When x is negative. When x is negative you have to push it in the positive direction. So this has to be positive. So control has to be positive.

So negative x positive control. Similarly positive x negative control. So what have we concluded from these three points? Control is large for small x in the positive direction control is negative in the negative direction control is positive. So what happens as I come closer and closer to the origin? You see what I have just drawn here exactly this. Here you big control big positive control big negative control got closer big positive bigger negative even closer very big positive very big negative.

So you can see what's happening. This cannot be a continuous control at all. Right. As you get closer to the origin control is exploding in the opposite direction. So I mean it's not even a very very scary looking example. I mean it doesn't look scary on the top of the just looking at it doesn't look that scary but it is a very very bad system that you can't design continuous controllers for.

Okay. So this is sort of the example. So this sort of a system does not satisfy a small control property. Yeah. Because even if you are close to the equilibrium you are not going to get this kind of property.

Impossible. You are going to get very very large controls. I mean infinite control if you get very close to the origin.

Yeah. Okay. Unbounded controls. Okay. So that's maybe one of the reasoning why you know you this seems like a reasonable assumption that if you are close to your equilibrium you should have should require less effort. Yeah. Nothing very very bad should be happening with the system.

So this is a very reasonable sort of a control continuity assumption. Okay. Okay. So if you do have such a small control assumption then you have this very very strong result called the Archtime Sontag theorem and this happens to be a constructive result. In fact one of the few constructive ways of coming up with a control law if you are given a control Lyapunov function. Okay. So what does it say? This is called the Archtime Sontag theorem or the

Archtime Sontag I mean the corresponding control law is called the Archtime Sontag universal formula.

Yeah. What does it do? It says if you have a control affine system just like we saw and if there exists a control Lyapunov function for the control affine system then the system admits the small control property if and only if it admits an almost C infinity stabilizer with u0 equal to u bar. Okay. So very strong result.

Why it's a very strong result? First of all it's an if and only if result. Yeah. In typical mathematics and applied mathematics if and only if results are considered very strong result because they are very tight. Yeah. It's like this implies that and that implies this.

So you can't have one without this. It's a very tight result. Yeah. So it says basically that the assumptions that you made are the least required for you to have a control like this. Yeah. So this is these are good. We have considered very good results.

The other thing is because this is constructive we look at it later. It actually gives you an expression for the control. Okay. Now the only sort of fine point to see is that it says that it admits an almost C infinity stabilizer.

Okay. You already know C infinity would be smooth. Okay. And you know what is a stabilizer? Stabilizer just means that it the control will make you asymptotically converge to the origin. Okay. So that would be a stabilizer. But what is the almost? The almost means that all the nice properties are still in a perforated neighborhood of the origin.

Origin is not included. Okay. All you can get is continuity at the origin. Okay. This is what you will get out of this result. Okay. Out of this result also this is what you will get. I mean for systems like this with no small control property this doesn't exist.

And you already seen that it is a very tight result. So no small control property, no all continuous stabilizer at the origin. Okay. So if you don't have small control property there is no possibility which is sort of evident also.

Right. Usually control flips direction at the equilibrium. Right. Sort of very natural, very intuitive that the control flips direction at the origin. Because if you are on one side of the equilibrium you are pushing it this way, other side pushing in that way. So very natural that if you are on the left you are pushing right, on the right pushing left.

Okay. So I can think of this for aero mechanical systems with you know position, velocity as states. But same thing can be thought of in electrical biological systems also. So one side of the equilibrium push one way, other side push other way. Yeah. So this actually gives you a way of constructing an almost C infinity stabilizer which means that it is smooth everywhere but at the origin where it is continuous.

Okay. So that's what you can sort of achieve with this Arstein Sontag formula. I will just show you what the formula is very quickly and then we will end. So in order to give the control they use the of course the whatever elements are given to us which is the vector fields the control Lyapunov function. So you construct an A of X which is here which is coming from del V del X F0 which is the drift vector field.

Yeah. Then you have a BX which is basically the vector consisting of all the control vector fields. Okay. So as you can see this will be a matrix. What will be the dimension of this guy? What do you think is the dimension of AX? How many states? N states.

Okay. So what is the dimension of del V del X? No. Del V del X. V is what? What is the dimension of V? I mean V is what? V is scalar value. Okay. So partial of V with respect to X what is the dimension? Yeah.

You can say n cross 1 or typical convention is to say 1 cross n. Yeah. You think of partials as row vectors. Del V you write it as del V del X1, del V del X2, del V del Xn.

Okay. Typically this would be your 1 cross n vector. Okay. What is the dimension of F0? n cross 1. Right. It is a just a vector field n cross 1.

So dimension of A? 1. A is a scalar. Excellent. Similarly, dimension of del V del X Fi X1. Okay. So this is actually a vector then.

Right. This is actually a vector. Okay. What is the dimension of the vector? It is m dimensional vector. Okay.

An m dimensional vector. Okay. Great. So this is what is the Einstein-Sontag-Universar formula for the control. Okay. Very cool control. Called a universal formula or many people just call the universal formula or the Einstein-Sontag universal formula. But it is one of the few formulae that gives you directly a way of constructing a control if you have a control Lyapunov function.

Okay. This will always work. This control is C infinity everywhere but at the origin where it is continuous. So this will always work. You can take any system, any robotic system, any aeromechanical system, any electrical, any biological system with a model and a control Lyapunov function.

This will give you a stabilizing control. Okay. And that is something super strong. Right. For any arbitrary system once you have a V you are coming up with a, you basically have a formula. Okay. So hardly do you, I mean I don't think most of you would know of any non-linear control formulas.

Right. That you can just plug and play. Right. Like this. Okay. Looks very very ugly actually for lack of another word. But it is actually quite nice and it behaves very well. It is a well behaved controller. Yeah.

Again because it is C infinity everywhere and you know it is continuous at the origin. It is a very nicely behaved controller. We will discuss it next time. So you can see I am using A. I am using norm of B and B itself.

So you have a you know. So basically if you look at this expression B is what defines the direction of the control direction. The control direction is defined by B. Okay. And you see there are two cases here.

When B is zero and when B is non-zero. Why? B is zero corresponds to del V del x Fi equal to zero for all i. That is the B equal to zero. That is how B is defined. See if del V del x Fi is equal to zero for all i there is no point in applying a control because control has no effect anyway.

So what is the point? Apply zero control. Okay. But when if B is non-zero if you remember what were we doing? We were inverting that particular term. Right. If we said that the second if del V del x F2 x is non-zero then we just inverted del V del x F2 and created a controller.

Okay. So this is too basic because you don't know at which instant which one is going to be non-zero. So this actually generalizes that idea.

Okay. You don't know which one is non-zero for that particular x. So depending on whichever one is non-zero this will work always. Okay. So that's the idea. Yeah. This is the universal formula and we will stop here. Okay. Alright. Thank you.