Nonlinear Control Design

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Week 1: Lecture 3 : Existence and uniqueness of solutions

So, we were looking at these models, I mean there are couple more models that we will sort of you know look at. This is the anti-lock braking system model, again a very mechanical system model. So here you look at the car as a quarter car, what is called a quarter car model. These are all very standard terminology in automobile system. So you use the quarter car model to you know do a lot of design, suspension, braking. So you can do a lot of design with just the quarter car models and you expect them to work well in the real system also.

So the quarter car model looks something like this. Basically it's you have again the state variables are this omega and V which is the angular's vehicle speed and the longitudinal vehicle speed. Basically since it's a quarter car model it has just one V. So you just have the angular vehicle speed and the longitudinal vehicle speed.

And then you have the factors that are affecting these are your braking torque. This is sort of your I would say your control. And then you have this Fx which is the longitudinal tyre road contact force. So this is something that actually depends on your tyre and so many other factors weight and so on and so forth. So this is something you have to sort of arrive at for different systems differently.

Then you of course have J, M and R which are basically your inertia V of the wheel. M is the mass, R is basically the wheel radius. And this Fx itself as you know the lateral force always depends on the normal force. So it's basically the normal force multiplied by some coefficient. So you can see that there is actually no nonlinearity here.

It doesn't look like there is any nonlinearity here. Nonlinearity actually enters here. This depends on so basically you have the Fz and then you have the mu which depends on these lambda, beta, tau and theta R where lambda is a longitudinal slip which is you start seeing some nonlinearities here. And then you have the wheel side slip angle and you have some road dependent parameters. So basically this is sort of the model.

So no nonlinearity here but you start to see nonlinearity because of this expression and this expression. So again even for this quarter chord model you can see that it's a relatively complicated model. So you might need further simplifications if you are actually trying to do control design. But it's not impossible to start from here itself. Because this is sort of in this particular scenario it looks like you can sort of cancel the nonlinearity also.

So there is a possibility that you can work with even this model and do something interesting. So those were all the examples that we wanted to talk about. Now one of the key things, this is where the regularity aspect of F comes in is existence and uniqueness of solutions. So this is something that I always talk about in my nonlinear control course. This is actually material from an ODE course.

But since we are talking of a nonlinear system we sort of at least touch upon it. I am not going to prove anything here. But I am just going to give you a feel of how things can go wrong if your function is not appropriate. So which is why more often than not we assume things like smoothness when we give our results. Because if it is not the case then you have to very carefully evaluate your system dynamics to see if you can even use the analysis methods that we are talking about.

Designing control is a much much further aspect. The question is whether you can even talk about analysis in the classical sense that we talk about in this course. You might need more advanced notions. So the question is when do solutions exist and are unique. So both of these are important for us.

You can see this small little graphic that I sort of made here. And on the x axis here is basically time. And on the y axis are the system trajectories. So this is the state. But we don't say state because once we solve it we say trajectories.

So this is the solution for this system given some initial condition. So you can see that I have started with sort of two different initial conditions here. You can see that it is here and here. But somehow it seems like once I evolve these from these initial conditions there is a meeting point here. This is an example of non uniqueness in solutions.

And this is not that uncommon by the way. So this is an example of non uniqueness in solutions. Because at this point let's forget everything that happened before. But if I initialize this system at this point.

Yes please. Can't they be the same states at same state at different initial conditions? Absolutely. So whenever so your question is are they the same states at different initial conditions. Yes there is no other state. See look at this. There is only x as the state.

Okay all I am saying is state is just an abstraction. It is just a representation. I am saying you know current is the state. Angular velocity, angular position is the state. And this is an abstraction of it's just words right.

But when I want to solve a differential equation I need initial conditions. So the question is what initial conditions? And depending on my system at that particular instant my initial conditions may be very different. So initial condition may be different, initial time may be different. So you need both. So this is in fact for the same state I give two different initial conditions.

And this is the sort of evolution I get. I mean this is the solution I get starting at this initial time t0. Now if I look at this as my initial time say. If I look at t1 as my initial time. What happens? The system can go in either side, in either direction.

So there is no real clarity on where the system is going. So it is not even clear which way the system is evolving for me to be able to talk about equilibrium and stability and things like that. So this is one of the big concerns if you are going to talk about notions like stability in a sort of classical sense that we do. Let's look at examples. I like these examples because they are very illustrative.

So this is an example of a system where of non-existence of solutions. So it is a very simple system. It is just x dot is equal to x cube and I am giving my initial time at 0 and initial condition is just x0. It's not very difficult.

All of you can integrate it. This is what the solution comes out to be. What goes wrong? If I have non-zero initial condition say x0 is non-zero then it is very evident that the solution will exist only until this is positive. As soon as this quantity here becomes negative it is imaginary. So there is no solution to the system. So beyond a certain time and you can see this is dictated by how big time is because x0 is fixed everything is fixed here except for time.

So as time becomes large in fact goes beyond this yeah things get messed up. In fact very odd thing happens at exactly at this time the solution is actually infinity right exactly at this time solution is infinity right because this is exactly going to be 0. So you know I have an sort of an infinite jump. So you can imagine that as t gets closer to this value you know your solutions are going up up up and becoming you know very large numbers. But beyond this also funny things happen right.

It's not like it's not a one point escape time. It's not like only at t equal to 1 over 2x0 square I have a problem. No beyond 1 over 2x0 squared this is negative. So it's imaginary.

So there is no solution. So this is you know an example of non-existence of solutions. So this is these kind of systems why are they a problem? Why do you think they are a problem? Why? Why don't I like them? As control guys why do we have issues with such systems? I mean the non-existence the cases of non-uniqueness can still be dealt with in some contexts by the way. But non-existence of solutions is you know a very difficult absolute no go almost. See the problem is it's a you are talking about non-linear systems ok. And at least and in this particular course we are still talking about asymptotic results right that is controls which sort of give you convergence after a large time.

You may be happy with whatever happens in 10 seconds but the theoretical result tells you that at infinity you converge to zero ok. Now modern day control theory of course has finite time control and this and that ok. But again there is fixed time control but then all a lot of this such controls are non-smooth controllers yeah. And sometimes also discontinuous controls and so not discontinuous in the first derivative but discontinuous in second order derivative and things like that yeah. And these such controllers typically stress your actuator a lot yeah.

If you are talking mechanical actuators then these are very very high frequency controllers yeah. So but if you talk smooth controllers like we do here in this course then you are talking about infinite time that you need. But here I have very very limited time to do anything yeah. There is no guarantee that I would have achieved anything in this time ok.

So these are sort of issues. Again this is not a control system let me be clear. This is not a control system but I am saying even if you pose a control problem you know I add a control here yeah. It's still a problem because when I apply zero control it escapes yeah. If it escapes then there is no solution yeah. So I mean yeah I mean it's very difficult to work with such systems.

So we typically do not like systems which have non-existence beyond a certain time. These are definitely tough tough tough systems to handle ok alright. Homework 1. This is how the homeworks look ok.

So this is the homework 1 I mean yeah. So I want you to give one more example ok things like that alright. Example of non-uniqueness yeah I just flipped it so it was x x cubed I made it x one time ok. So this is an example of a non-unique solution and I give you an initial condition at zero time as x zero equal to zero. I start with zero initial time. So one thing should be obvious to you that this is a solution yeah.

This remaining at zero for all time is a solution yes. But then I can also integrate this. I can actually integrate this is also not difficult to integrate and you will find that the integral is this yeah. I mean it's it's not difficult to verify because I can take x dot and this comes out to be this yeah. Now this is also a solution right because because this is zero at zero notice.

How do you say anything is a solution of a differential equation? It has to match at initial condition and it has to satisfy the differential equation. These are the only two requirements match at initial condition. So this matches at initial condition. This satisfies the differential equation. Why if I take derivative of this guy still zero right and the right hand side is also zero.

So this is a solution yes. Obviously this guy if I take a derivative satisfies this guy yeah because I obtained this by integrating this obviously satisfy and at initial condition at zero time this is zero right. Satisfies initial condition satisfies differential equation yeah. But see these are different aren't they? If I plot it what will happen? If I make it like this with time here I don't know I really don't know how to look. I know that the zero solution will look like this yeah and this guy will look something like this I don't know.

I really don't know. I think it will look like this. That's my guess. I don't know yeah. Better if you plot it yeah. Whatever may not look may be other way round but just you can just imagine that this guy looks like this.

The point is away from zero this is non-zero. That's the key thing to remember right. Away from zero this is non-zero but this is not yeah. So at this point I have an issue yeah and notice this point is not any point. This is the equilibrium of this system.

I hope all of you know what is an equilibrium right. Wherever the right hand side is zero anyway we will formally talk about it but wherever the right hand side is zero is the equilibrium of the system yeah. So zero is an equilibrium okay. So this is sort of an issue right. You don't know especially at equilibrium you see that the solutions are going at two different directions. Again an issue right but these are sort of not impossible to work with especially in the modern theory of sliding mode control and finite time, fixed time control.

These sort of non-unique systems with non-unique solutions are dealt with. How you deal with them is you talk about stability of all possible solutions okay. Instead of talking about you know in our kind of theory in the classical theory that we talk about, we talk about stability of an equilibrium. Here in this more modern theory I would say we talk about stability of all equations, all solutions okay. So it is a slightly different notion yeah of how you work with things but interestingly things don't change too much yeah.

So because typically sliding mode control like I said discontinuous or non-smooth results and non-uniqueness and there you have to have notions of what is the kind of stability we are talking about okay alright good. Any questions? Alright. So again homework 2. Give one more example. So this is the sort of result that we assume more or less for this course.

For this course we assume that our function f which governs the state space model without the control that is has an existence and uniqueness property yeah. What is it? If you have the system x dot is f t x, x is in Rn, x t0 x0. We assume that f is piecewise continuous in time and satisfies a global Lipschitz condition in x in the states.

Looks like this. This is what is the global Lipschitz condition. This is actually a regularity condition, sort of a smoothness condition yeah. You can see that if you have this kind of a property it is guaranteeing differentiability okay. So this is something more than differentiability yeah. So this is f (t, x) minus f(t, y) in the norm has to be less than equal to L norm x minus y.

Why is it global? Because this holds for all x y in Rn okay. I am not saying x y in some ball or x y in some small region around the origin or anything like that. I am assuming global Lipschitz which means that this holds for all x y in Rn okay. And some constant L positive okay. Then for any initial condition x0 the above ODE has a unique solution for all time okay.

So this is the standard result for existence and uniqueness. You can find it in Khaal in the Khaalel's book yeah. Maybe in Vidyasagar's book I don't know but Khaalel's book for sure yeah. So very standard result. This is the sort of result we assume okay for this course yeah. Because we don't like we don't want our trajectories to intersect and all that yeah.

No intersection of trajectories okay. We don't want trajectories to intersect, we don't want trajectories to vanish and not exist. Neither is cool with us. So we assume this very very if you may harsh condition yeah okay. So again lot of systems may not satisfy this.

In those you will have to deal with them as very special cases yeah. Impulse systems I mean if I am thinking of dropping a ball every time it hits the ground you lost yeah lost the smoothness yeah. So lot of simple systems that you can think about may not satisfy this yeah. In those cases you just have to why this dropping ball thing is also an important example is because all this biped, quadruped, locomotion is all based on this yeah. There is an impulse okay. Then how do you deal with it? So you sort of do some special case sort of a situation there yeah.

Anyway the lot of time the control is also based on some approximation there yeah. So in those cases you don't think about too carefully about existence and things like that because physically somehow you know that solutions exist yeah. So you are not too worried that you applying some control will make the solution not exist. So to think like that so those are again special cases yeah.

So whatever we do here we assume this condition is satisfied alright. Any questions? No? Alright I think we will stop here. Thank you.