Nonlinear Control Design

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Week 5 : Lecture 28 : La Salle's Invariance Principle: Part 6

Welcome to another session of control of non-linear dynamical systems. Well, we are into the lectures 8 to 10 as you can see in the label. Well, you cannot see in the label, but I can see it. Now we have been talking about LaSalle invariance principle and I believe you have also seen some examples in the tutorial. So hopefully you have a little bit more clarity on how to use this. I want to revisit the example because there is a small error.

But anyway, let's again go back and do this example and then I will go back to the proof and we will go through the proof. Anyway, the proof was not completed. So we will try to finish it today and then go ahead with whatever the rest of the material. So this was a spring mass damper example and we were trying to use not the general LaSalle invariance principle, but the Barbashim-Trosovsky LaSalle which is the theorem that gives you stability of the zero equilibrium.

So this is the asymptotic stability result. This is what we are trying to use in this example. So this is the very simple linear system corresponding to this guy. Again the picture and the constants don't correspond, but it's very easy to derive this. One is K by m, another is C by m.

So very standard and simple dynamics. And we choose again a very standard candidate Lyapunov function. In fact, this does turn out to be a candidate Lyapunov function. Remember that for applying the Barbashim-Trosovsky LaSalle theorem, we do need a candidate Lyapunov function. That is it has to be C1 and it has to be positive definite.

So this is in fact C1 and also radially unbounded. So it's a linear system. So obviously like you remember, global and global are all the same. And then when you compute the V dot, it will turn out to be minus K2X2 squared, which is just negative semi definite. Now of course, we want to apply the Barbashim-Trosovsky LaSalle.

We know that this spring mass damper system, if you just look at this system and you know that if you just leave this mass, you pull it and leave it anywhere, it's going to come to a stop unless you apply some external force. So you know that this is in fact an asymptotically stable system. So how do we prove that? We use the theorem that we have and we first define the set E, which is the set which has V dot equal to 0. So remember that for applying this sort of a theorem, we don't care about the invariant set and so on and so forth. If you remember this theorem does not require us to construct the omega set and so on.

So we don't worry about the omega set at all. It will seem like we are working with the entire domain. But you also understand very well, I hope that we can construct the omega set just by using this V function itself. We have done this in the pendulum example. So what is the set E? It is the set where x2 is 0 and x1 is arbitrary.

Then there is a sort of an incorrect statement here. So as always we start by assuming that E itself is the invariant set. And for E to remain invariant, unfortunately it is stated here that we need both x1 dot and x2 dot to be 0. But that is not required. x1 dot need not be 0 obviously because you can have anything in the first coordinate.

Therefore even if it changes, we don't care. It is whatever. Anything is allowed here because this does not contribute to V dot being 0. So this is not required. That is why I have now crossed it out.

Thank you for pointing it out. And we only need x2 dot to remain at 0. Because if x2 dot is non-zero, then I move out of the 0 in the x2 coordinate which is a problem. This is not okay. So in order to have x2 dot to be exactly 0, we start looking at the dynamics.

If you want x2 dot to be exactly 0 and you already have x2 to be 0 because the definition of E is that x2 is 0. Then the only way for x2 dot to be 0 is for x1 to be 0. And k1, k2 are strictly positive. So x1 has to be 0. So that is the idea.

So now what have we shown? We have shown that the only way for us to have invariance is if both x1 and x2 are 0. So this is what becomes our invariant set M, the largest invariant set M inside E. And because this largest invariant set in this case contains only the origin, the origin is asymptotically stable. Okay. Remember we cannot say anything about exponential stability etc.

from LaSalle invariance. That is not possible. Yeah. And in general nonlinear systems anyway we do not target exponential stability. That is a pretty strong result.

But again I mean these are all what I am, this course is more I would say still classical. In the sense this material is all classical material. I am not talking about modern material. In modern material you have, you can actually do fixed time, finite time controllers, fixed time controllers. So obviously things have moved ahead significantly.

First of all we are only covering classical material A. That is not the only thing. If you want to do things like fixed time, finite time, these kind of controllers then your controller will become non-smooth. So you can have jitters, sharp changes in your control. So depending on your application, on your actuator, ability of your actuator to reproduce really fast changes.

For example if you ask your motor to change really RPMs very quickly. Your motor still may be easy. If you go to a gas thruster very difficult. So depending on your actuator, depending on your actuator it is your call. I mean if you are saying change in voltages sure maybe you can whatever you can have a very quickly, quick acting potentiometer circuit.

A digital circuit can act at a pretty quick rate I mean 100 hertz and so on. So depending on your requirement you can potentially achieve this kind of control or finite fixed time stability or not. So in this course we are only talking about smooth controllers. So that is also why you get only asymptotic results and sometimes exponential results.

Alright great. Just wanted to correct this. So this is an error. I fixed that now. Now let us go back to the proof. I am going to start from the beginning again.

We have talked about all the terms involved. So I am not going to redefine these. And this is the proof of the general LaSalle invariance. The proof of Babasheen Krasovskiy LaSalle is a subset of this. Obviously if I hope it is very easy for you to see that if you satisfy this then you definitely satisfy this.

Yeah. Assumptions, all these assumptions here definitely imply this. Yeah I hope it is obvious because we just use the same method. The only thing you have to do to apply this result was to actually define a omega. But we have already done that in our pendulum example. We use the V itself to define an omega.

Right. So once I have this kind of a condition to be satisfied this is definitely satisfied. And when you get here that zero is the only invariant set then zero becomes asymptotically stable. You converge to zero.

Right. By this result. The only thing remember let us be careful here. The statement of LaSalle invariance principle does not talk about stability. Does not say origin is stable.

Ok. Here you do say that. If you say asymptotic stability you have stability and convergence. Both. Here you don't say anything about stability of origin. You just talk about convergence to a limit set. Why because you cannot talk about stability of origin in the general case.

Again Van der Pol oscillator. When you get confused think Van der Pol oscillator. Because a limit cycle behavior origin is actually not stable. Right. So you can't talk about stability in general. But when you have these kinds of assumptions happening.

That you start with a positive definite V and you have a negative semi definite V dot. Ok. Just by Lyapunov theorem you have stability. Right. Because I started with V positive definite and V dot was negative semi definite.

So just by the Lyapunov theorem I have stability. Done. I have stability statement. Alright. And so when I go here I don't need to obtain stability from LaSalle invariance.

Stability is already done by Lyapunov theorem and convergence given by LaSalle invariance. Ok. So it should be obvious to you that these results if you have these satisfied these are much stronger requirements.

Much stronger requirements than this. Ok. So I am not going to actually prove it. But it is pretty straight forward I think. Ok. Great. So we only prove the more general case that is the LaSalle invariance principle.

Absolutely. Not for every V. For every V with V dot less than equal to zero. Recall the pendulum example. How did we do it? We use the fact that V dot is non-increasing over time. Therefore whatever initial value you start at you remain below that value.

That gives you a invariant set omega. Compact invariant set omega. Done. Ok. So these two together are required.

Not just any positive definite V will not work. Ok. Alright. So how did we go about the proof? We started by saying omega is close in bond. We just started by looking at all the assumptions. Ok.

So omega was compact which implies closed and bounded. Omega is invariant. Ok. So if I start in omega my entire trajectory remains in omega for all time beyond initial time. Ok. Now this implies that if I start in omega my trajectories are bounded.

Right. So the entire LaSalle invariance postulates that you start in omega. Ok. So that is required. Once I start in omega, omega is compact.

Therefore all trajectories are bounded. Ok. And once you have this bounded trajectories Vidya cycle gives really nice two results. Three results in fact. And that is what we use pretty much to complete our proof.

We are not doing anything honestly. Ok. The first result says that if you have bounded trajectories then the limit set denoted as such. Ok. Notice it is indexed with x0.

It depends on the initial condition. Right. The limit set is non-empty, closed and bounded. Ok. What is the limit set? Limit set is where all the points go towards.

Ok. This is essentially the definition of the limit set. That is the set towards which all the trajectories will go eventually. Ok. Eventually they will go to some point in the limit set. Ok. And we are now saying by this Vidya cycle's result that it is non-empty, closed and bounded.

Ok. We have not yet connected omega bar and omega. Ok. But I think we sort of gave it a thought and we realized that omega bar has to be a subset of omega. Ok. Anyway. Yeah. Because if your trajectories are starting there and remaining there, starting in omega, remaining in omega, therefore the limit set also has to be within omega only and be outside.

Seems ridiculous. Ok. So, limit set is non-empty, closed and bounded and inside omega. Second result again by boundedness of trajectories. You have that all your states, all your solutions converge to the limit set. This is more just by the definition of limit set itself.

And secondly, omega bar is also an invariant set. Ok. Alright. So, this is again something that we sort of, you know, sort of understood by these kind of examples. Yeah. And I think I made you write a few points which I asked all of you to memorize. Those of you who have not written these points, write it from your friends and then memorize.

Ok. Because there is no way we are going to prove all this and it is going to get etched in your memory or anything. Yeah. You can look at the proof if you want for all this.

But yeah. So, limit set is always closed. Ok. Limit set is always closed, limit set is invariant. You start in the limit set, you remain in the limit set.

Ok. Great. So, you have these three very very nice results. Yeah. Courtesy Vidya Sagar. Right. Well, I mean he may not have come with the results on his own but whatever.

For us, courtesy Vidya Sagar. Ok. Alright. Great. Great. So, we want to connect this omega bar with the set E and so on and so forth.

Ok. This is our plan. Yeah. Because LaSalle invariance gives these elements. It gives us the set, once you start with the set omega, it gives you an E and then gives you an M. So, what we want to do is, we want to connect this omega bar to this E and M. That will be our aim. How do these sets compare? Ok.

So, then we sort of invoke what is called a monotone convergence theorem. I have not stated it in this course but we use it regularly in adaptive control. So, this is a very standard result that is required in adaptive control. Here we saw probably require it only once here.

But it basically says that if a function is lower bounded and non-increasing. Ok. Function is lower bounded, non-increasing. Yeah. Then limit as t goes to infinity of the function exists.

Ok. You can state it the other way around also, the monotone convergence theorem. If the function is upper bounded and non-decreasing, then also it has a limit as t goes to infinity.

Ok. This limit need not be zero unlike what I have stated here in this notes.

So, I have cancelled and written C. Ok. This is basically the monotone convergence theorem. Ok. So, function and so in our case the function V is in fact lower bounded.

We took it to be positive definite. It is a candidate Lyapunov function. Yeah. And it is non-increasing. Right.

Because V dot is less than equal to zero. Right. As a function of time it is non-increasing.

Can be flat or going down. Flat or going down. Can't be going up. Only two possibilities. Ok. Great. So, it has a limit as t goes to infinity. Whatever with this limit is we don't care.

Now, cool things start to happen because of the fact that V of t is constant. Yeah. V can be seen as a function of time. Right. We already know this. This by plugging in the solutions V becomes a function of time.

That is how we take the derivative also. I mean that sort of the notion of taking the derivative although the derivative is something we have defined as the directional derivative. Right. But this is the notion that V is a function of time actually.

Yeah. So, when you plug in the solutions. The only problem with this very very theoretically intense folks will tell you that when you plug in the solution you are plugging in the initial condition. So it is not just a function of time but also a function of initial conditions.

Ok. So, whatever result you get is not uniform with respect to all initial conditions. Ok. Therefore if you notice this LaSalle in when you look at this limit set X0 dependence is clearly stated here. Ok. This is for this particular reason because whenever you plug in all this V of X of t here I didn't write it we are using the short hand remember but this is actually plugging in the solution.

Right. I mean it is plugging in this gate. Yeah. So, therefore there is an X0 dependence here. Ok. We can't get rid of that.

Alright. Ok. Now what we plan to do is we plan to show that omega bar is in fact inside the set E. Ok. How do we do that? We already know that omega bar is inside omega. Very easy to argue. We already argued it in fact. Now what did we do? This is where we played a fun trick.

It seemed to be difficult to digest but it does happen. We take any point arbitrary in the limit set. Ok. Any arbitrary point in the limit set. Now we know by definition of a limit point because P is a limit point that there is a time sequence such that the solutions converge to P. Ok. By time sequence I mean this Ti is not like continuous.

It is you can take any time sequence 1, 2, 3, Ti can be, T1 can be 1, T2 can be 2 or T1 can be 1, T2 can be 1.2, T3 can be 1.4 whatever. It is a time sequence. Does not have to be any.

The only thing is i has to go to infinity, any sequence and Ti has to go to infinity as i goes to infinity. Ok. Therefore, so what we are saying is that as time becomes really large in this sequence you do converge to this limit point. Ok. This is the definition of the limit point.

Ok. So, this is the result it might be to grasp but we have seen these examples. Right. I mean we saw sequences like half 1, half 1, half 1. Where if I take Ti equal to 1, 2, 3, 4 this is the problem. Does not converge to anything because it oscillates between half and 1.

Right. But if I take Ti equal to 1, 3, 5, half, 2, 4, 6, 1. Alright. So, there is always a possibility of multiple limit points. Ok. Not that unusual. It is not that unusual.

And although I have given discrete examples even continuous examples are not difficult to construct.

Ok. In this way. Ok. Alright. Great. So, P is a limit point. Ok. And what now I start playing with continuity. Ok. What is this? I, let's see, let's see.

I write this expression first. First I write this expression. Let me write this expression first. Ok.

And I write it as I goes to infinity V of X Ti. Ok. Alright. Alright. I hope you see that this quantity is actually just C. Alright.

Ok. Why? Because Ti is going to infinity as I goes to infinity. Ok. So, I might just even write, just write it like this. Does not matter. Yeah. Because however it goes to infinity I don't care in whichever direction. But the point is Ti is going to infinity because I going to infinity which means that this comes to force.

That limit exists and it is some constant value C. Now, by continuity of the function V and also X. X is also continuous.

The solutions are continuous. Right. V is continuous. So, I am free to move this limit inside.

That is what I do. Move this limit inside. Right. And this quantity just inside this is just P. Ok. So what I have is V of P. And so I have just proved it V of P is equal to C. Now I did not, I did not you know have any bias towards any particular point or something in the limit set. I took an arbitrary limit point in the limit set.

Yeah. Therefore, if I had chosen any other point also, yeah, P bar, nothing would have changed. Right. The analysis doesn't change. Right. So, if it seems very very unintuitive or funny or odd, all the limit points in this limit set map to one value through V.

And that's this value C. Ok. Ok. So, I mean if I was to write in sort of text, it would, I would say that omega bar or the limit set is a level set.

Of V. Yeah. For those of you who know what level sets are. Level set is basically any V inverse C. V inverse C is the level set. Ok. So, omega bar not is but belongs to a level set actually.

Whatever. Belongs to a level set. So essentially what this is saying is that V of omega bar X0 is equal to this constant. Yeah. This is a notation by the way.

This is notation. You can't actually plug in a set inside a function. Ok. You have to plug in points from that set. And this notation, when you say V of a set, you mean you plug in points, all the points from the set.

So if you plug in any point from the set omega bar X0, you always get C. Ok. This is very cool then. Yeah. What will I do? I start with, I take any trajectory.

Yeah. X star inside this. Yeah. Basically what I do is I take any trajectory X star with initial condition here. Ok. And then I already know omega bar is invariant.

Right. In fact I should not have used, I have already used X0, right. Ok. I will use another initial condition. Doesn't matter actually. Ok. I start with some initial condition in omega bar.

Ok. That's all. Now if I start in omega bar, I know I will remain in omega bar because omega bar itself is an invariant set. Right. And if I remain in omega bar, I know for this entire trajectory VX star.

Yeah. V of X star of T is zero. Therefore the derivative along this solution that is V dot of X star, sorry V of X star is C. I am sorry. V of X star is in fact C. So therefore V dot of X star is actually equal to zero.

Yeah. Because along this entire trajectory in this limit set, if you think about it as I move here. Yeah. As I move along this omega bar set, my V doesn't change at all. Therefore V dot is actually equal to zero.

Ok. V dot is actually equal to zero. So what have I just proved? That on the set omega bar, on the set omega bar and you can just conclude it by this only. Forget all this starting trajectory and all that. You don't have to worry about starting trajectory. Just by this expression, you know that V dot of omega bar is going to be zero. Because V dot is, V is remaining constant on this entire set.

So V dot is zero on this entire set which means then any trajectory in that set and there are trajectories in that set. So the only purpose of seeing this sentence is to indicate that there are trajectories in the set.

Right. Because it is an invariant set. If you start there, you remain there. Ok. Therefore V dot is zero and V dot equal to zero describes what set? E. E. Right. So I can't say that, so therefore I can't say that omega bar is the entire E set or something but I definitely know omega bar is inside E.

Ok. That should be very obvious that omega bar is inside E because V does not change in omega bar. V does not change in E. Therefore omega bar has to be a subset of E. It may be the same size but it can't be bigger because E is the set where V dot is equal to zero in all of omega.

So therefore E is the largest possible set where V is, V dot is zero. Ok. So omega bar is definitely inside E. Ok. Excellent. Now then things are very straightforward.

After that it is just one sentence. Omega bar is invariant. Ok. So omega bar is, it is just, it doesn't only have the property that V dot of omega bar is zero. Ok. It also has a property that it is invariant. E does not have this property by the way.

We have already seen by examples that E itself does not have this property. Ok. But which set has this property? M has this property. That it is invariant and it is contained in E. So omega bar also is inside E also invariant.

M is also invariant but we already said that M is the largest invariant set. Yeah. So we said that M is the largest invariant set. Therefore omega bar has to be contained in M also.

Ok. Cannot be larger than M because if it was then omega bar would be the largest invariant set. M would mean nothing. Ok. So M is the largest invariant set. So omega bar is in M and therefore in E also.

Ok. So now what have we shown? So we already know that all trajectories are going to go. Yeah. Whenever you start inside omega you are going to go to omega bar. Right? And omega bar is inside M. Right? So LaSalle invariance is proved because I started in the set omega, the larger invariant set and I have proved that I go to omega bar which is a set which is inside M.

So therefore we have proved that we go to M. Ok. Now remember the interesting thing here is we never proved that omega bar is equal to M. So we proved that omega bar is not equal to E in most cases because E is not invariant. Yeah? Almost invariably the set E that you get in most examples will never be invariant.

You have to hunt for an invariant set inside E. Ok. We have done that in a couple of examples.

I am hoping that you have seen some more examples of the same. But that is the pretty standard situation. Ok. But omega bar could be equal to M. We have not stated it though. Ok. M could have points which are beyond omega bar by the way. The only thing we had to prove was that we converge to the set M.

Not all of the set M. We are not saying surjectivity or onto or we are not saying anything about those properties.

We are just saying that we converge to M. Ok. And that is proved by this. Ok. Absolutely. Absolutely. The whole sequence is here. I mean actually more or less here.

I can just add one more. Ok. So you start at omega and you actually converge to this guy. Obviously you cannot converge to random sets. You converge to the limit sets. Some trajectory will converge to limit sets. That is the purpose of defining limit sets.

The only thing we have done by our assumption is that we have proved that it is inside M. Why because see why? Why do we do all this? I mean one might ask. Yeah. You cannot actually compute limit sets of systems. You will never be able to do that. If I give you slightly more slightly complicated, I mean sure for this may be for this linear system grade.

But even for the pendulum example you will not be able to compute the solution at all to be able to compute limit sets. Ok. So there is no particular, I mean we want to find qualitative methods which let us conclude something about system behaviour asymptotically without actually computing solutions.

Ok. This is where linear systems, non-linear systems are a world apart. For linear systems everything can be solved. Ok. Sure there is lot of theory on you know overshoot control and whatever and you know you can do lot of cool things using transfer functions.

Sure great. No problem. I agree. But you can solve the system anyway. I can do the same in time domain. It does not look as elegant may be as it would do in transfer function domain doing overshoot minimization and things like that. But it is not impossible to do. Interestingly non-linear system will be impossible.

You cannot analytically solve the system at all. So you are relying on you know MATLAB or Python or whatever some ODE solvers numerically to even get a solution. Right. Which is not telling you anything about limit sets. No way.

If you just if I just give you an arbitrary non-linear system you can keep initializing it at many many different points. You may never be able to find a limit set. I mean how do you guess it. Ok. So these are the sure again linearization might help you a little bit but there is no guarantee.

Ok. You will miss a lot of potential limit points. Ok. So non-linear systems of course are more complicated but of course also offer more rich behavior. Yeah. And that is why we do not talk about you know we never say that I will compute omega bar. No you cannot compute omega bar. And so Lassa invariance just gives you a set M which is maybe a little bit conservative but good enough.

In fact you have seen that in the two examples that you have seen M is not even conservative right.

M is the limit set. Right. Here if you see M is exactly the equilibrium. So obviously the limit point. Yeah. The equilibrium is the limit point in this case. Yeah I hope you see that too. Yeah. And also for the pendulum case you can see that you had these two points which are again equilibrium. Ok. Which are again equilibrium. Alright. Ok. Ok. Thank you.