Nonlinear Control Design

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Week 5 : Lecture 26 : La Salle’s Invariance Principle: Part 4

We have now constructed the set E, ok. Now what? This is the slightly complicated part, everybody has trouble with using this part of the LaSalle invariance is to find the set M, ok and M has to be invariant and inside E and it is the largest invariant set, ok. Can't be anything, cannot be any invariant set larger than this, ok. Now how do we do this is, how do we usually do this? I mean because I don't know how to find the largest and smallest, right. I mean it is not like an optimization problem that I find largest and smallest because you can see that these will be very complicated sets, this kind of optimization will be virtually impossible to do. Basically you are, you will not find an engine which will solve this optimization problem if I actually thought of finding the largest invariant set and all because you will have to initialize, you will have to take all initial conditions and try to do something, ok.

Not possible, not possible. So what I will do is, we, or what everybody does, not what I will do, yeah, what everybody does is assume E is invariant, ok. So we actually assume that E is itself an invariant set, ok. So this sort of helps us continue forward.

What does it mean if E is invariant? Ok. What do you need for E to be invariant? It means if I start inside E, I must remain inside E. But look at the interesting thing, what is the set E? Is this guy, ok. Alright, it is this guy. Now the first term is a continuous, could be a continuous quantity, right.

It can take any value between minus 2 pi to 2 pi. Notice, this guy can take any value between minus 2 pi to 2 pi. So I can't really conclude too much about this. But what we use is the fact that the second term, that is x2, has to be zero, is fixed. So for me to be invariant and to remain inside E, I need x2 to be identically zero for all E in this case.

I need x2 to be identically zero. And what does this mean? If something is zero forever or some constant forever. And all these are continuous, right. So x1, x2 is continuous because I have a differential equation, right. So obviously continuous, nice properties.

So what does it mean? If x2 is zero for all time or any constant for all time? No, not really, right. That would mean all states are zero. Only one state being zero is not equal. Just like you know that alpha comma zero is not an equilibrium, right. I hope you understand that alpha, any x1 is not the equilibrium.

Otherwise it would be this zero velocity, zero velocity, zero velocity, all these will be equilibrium. So this is not an equilibrium. No. What can I say? If any quantity is, if any function is the constant function, what do we know about the function? This is function of time, right. I mean although I write it like this, it is actually a function of time.

Then x2 dot is identically zero. Yeah, x2 dot has to be identically. That is the only way function can be a constant function. Alright. Which means what? What is x2 dot? Anybody remember? This minus k sin x1 minus c x2.

This has to be identically zero. All of this to make E invariant. All of this is to make the set E invariant. Without this happening, set E cannot be made invariant.

Okay. Now I already know that this guy is already going to zero, right. And the left hand side is already zero. So what do I know? What do I conclude? This also has to converge to zero. Not converge to zero, whatever.

Has to be zero. Okay. Has to be zero. And what do I know? Sin x1 is equal to zero at what? x1 equal to what? n pi. Right. So pi, I will not write 2 pi because after that it is repetitions.

Right. I will just write zero and pi. And you see zero and pi are both in our set omega. Right. Both in our set omega. So we started by assuming that E is the invariant set but to ensure that E is in fact invariant, I required sin x1 to be zero which gave me exactly two values.

Not all of E. Not all of E. Okay. Everything else is the same, right. Zero and pi. All other angles are the same.

You can write them if you want but yeah, you can write zero and pi. I know 2 pi is also there but it is all the same. So I am not including them. So what do I have? I have that the invariant set is what? M is equal to what now? What is the set M now? 0, 0 and pi, 0.

Okay. Alright. Just by starting with E being invariant, I have concluded that only these two points constitute the invariant set. Okay. So M is the invariant set, the largest invariant set. You cannot have anything more. Why? Because I started with E being invariant which is the largest set I had to work with.

From there I could only get out that these two points give invariance. And what are these points? These are exactly the this and this. So what does LaSalle invariance say? Because I have satisfied all the requirements of the LaSalle invariance, I constructed a compact invariant omega. Right? I had a V and a V positive semi-definite, V dot negative semi-definite in omega. In fact it is in D but in omega also.

Okay? And so and then I constructed the E and the M sets. So I have satisfied all the requirements of the LaSalle invariance principle. So LaSalle invariance principle says that if I start in this set omega, not anywhere in the domain, if I start in this set omega, okay, then I will converge to this. Okay? So interestingly what I have been able to prove is still not if you say everything. Because although there is no restriction on the angle, right, it says if you start in omega, okay? So I can start at any angle because minus 2 pi to 2 pi closed set includes all the angles.

No problem. Okay? And it says I will converge to either this point or this point. It does not specify which point. You can converge here or here. No problem.

But the velocity is restricted. There is a restriction in the velocity. Although again in reality you know what? I can start at any velocity theoretically and I will converge to an equilibrium, one of the two equilibriums for a pendulum. Okay? In fact I will converge to this equilibrium only. I will never converge to this equilibrium. If the velocity is anything but zero, I will converge to this equilibrium only.

Okay? So in fact any velocity is allowed in reality. But I have only been able to prove this much. Okay? So, there is a caveat. Okay? And even LaSalle invariance does not let us prove what we know in reality. But this is more than enough because I can choose any c.

If you give me velocity 1 e power 6, I will choose a c accordingly and tell you, yeah this will work. Yeah? Because all I have to do is c. C was nothing. C is just an artifact of our imagination.

So c is nothing. We created it. If you give me 1 e to the power 6, 1 e to the power 10 velocity for whatever reason you want to put a rocket thruster on the pendulum but you will, I can still guarantee that you will converge to the equilibrium by choosing a large enough c. Okay? But, yeah, in theory once you fix a c, your velocities are bounded. Yeah? Not unbounded velocities. Unbounded velocities are not okay.

Which is obviously fair enough. So, do you all understand how to use the LaSalle invariance? I have also asked the TAs to try some examples. I hope they try something other than the pendulum. Anyway, with LaSalle invariance, application of the general LaSalle invariance with multiple equilibriums. Okay? So, it is important that we follow this. Now I also want to state, if you have no other questions on this, I want to state the stability versions of these theorems.

Okay? So LaSalle invariance was obviously by LaSalle. Yeah? But then the stability versions of these theorems were almost parallely developed by Krasovsky Barbasheen. Okay? So therefore, I like to call it Krasovsky Barbasheen-LaSalle theorems. Some books call it Krasovsky-LaSalle, some call it Barbasheen-LaSalle, but whatever.

It is better to put all the names. So these are very close to LaSalle theorems, but they only deal with stability of the zero equilibrium. Okay? Here you have a general multi, multi stability, what you can call multi stability because multiple points you can reach a limit set. Yeah? Here you specialize to reaching the origin only. Yeah? And how do we do that? We no longer start with V which is only positive semi-definite. We actually start with a proper Lyapunov candidate.

Okay? This is now a proper Lyapunov candidate, positive definite in C1 and we just have semi-definiteness on the V dot. Such Lyapunov candidates are called non-strict Lyapunov functions. Okay? So for all x in D, notice for all x in D, not in any omega. You see that the omega is carefully missing here.

There is no omega here. Why? Because I know that I can construct omega just by this fact. We just did in the example. And then if I define the S similarly whatever we called E here is the S here. Just because it is a stability theorem, so I am using a different notation.

That is all. If you define the S as such and if x equal to zero is the only invariant trajectory in S, this is just wording that some texts use but this is the same as saying the set x equal to zero is the largest invariant set inside S. Then x equal to zero is asymptotically stable. Okay? On the other end, you can also state a global version which says that if now you don't have a domain, you have all of Rn. Okay? Again, negative semi-definite in all of Rn and you have the same things happening, then you have global asymptotic stability.

Okay? That's it. If the domain goes away, then you have global asymptotic stability. Okay? Notice all these results, LaSalle invariance, Barba Sheen, Krosoczky, LaSalle, all these are only for time invariant systems or autonomous systems. Okay? For time varying systems, the results are significantly more complicated. Okay? I mean not easy to apply these results. Okay? Because the notions of limit points and limit sets itself becomes very messy.

Okay? Alright. So, these are the stability versions of the theorems. Alright? Now, if you go back to our example, how would I do the stability version? I mean in fact, this is actually pretty straight forward. Yeah? What will I do? I will no longer take minus 2 pi to 2 pi. I will just take minus pi to pi. Because I only want the bottom equilibrium included in the set now.

Because I am looking to do stability of zero equilibrium. I will not include the top equilibrium in my set D. Okay? So, my set D will now contain only minus pi to pi. Okay? Anyway, this is something I will ask you to complete.

Yeah? But the rest of the steps are exactly the same. You still have a V. Okay? Notice that as soon as I choose x1 in minus pi to pi, this is positive definite. This we discussed.

Right? Earlier in the example. We did this example. Okay? If x1 is in the minus pi to pi range, not minus 2 pi to 2 pi range. In the minus pi to pi range, this is a positive definite function. Right? I hope you all agree. Yes? Because this function will not be zero anywhere but at 0,0.

The problem with the larger range is 2 pi is included. Then it is not positive definite. But if 2 pi is not included, so minus pi to pi. Then 0 is the only point, that is x1 equal to 0 is the only point where this can become 0.

Yeah? x1, x2 equal to 0. Nowhere else. Okay? So, the only difference is in the domain. Okay? So, it will become minus pi to pi cross R. After that, it is very straight forward. V is positive definite, V dot is negative semi definite in the domain.

Okay? And after that, the same analysis goes through it. This analysis is not changing. The set E or in this case the set S, whatever you can call it S. But this is going to be exactly the same. Right? Because V dot is exactly the same.

So this set is the same. If this set is the same, the set M is the same without this guy. Because minus pi to pi does not contain pi at all. So, minus pi to pi opens it. Remember, we took for this domain is minus pi comma pi. Okay? For applying the Barvashin-Krasovski-Lassalle, we took, we take this.

So this is not even part of this set. Okay? So M contains only the origin. M contains only the origin. Okay? All this, so nothing changed in the analysis. I just shortened my or reduced the size of my domain. And just by reducing the size of my domain, instead of applying the General Lassalle principle, I am applying the Barvashin-Krasovski-Lassalle.

Why? Because by shrinking the domain, I removed this equilibrium. Minus pi to pi does not contain this guy. Yeah? Because I excluded pi. Okay? That's all. So by shrinking, so what am I doing by saying minus pi to pi? It is starting from here and here.

So it's like, it goes from here all the way to slightly here. That's it. Minus pi to pi is just everything but that vertical, upward vertical line. Okay? So I have essentially taken everything but that top vertical line and so I have skipped the second equilibrium. Once I skip the second equilibrium, there is only one equilibrium. Okay? And remember, I always told you, when you have one equilibrium, only then you can talk about stability or global stability. If you have multiple equilibria, then you have to think Lassalle type ideas.

So that's all. So I have shrunk the domain so that I have one equilibrium in this domain and then applying Barvashin-Korsowski-Lassalle is very easy because the set E remains the same and the set M contains only the zero equilibrium and that proves that zero is asymptotically stable. Not global because of this restriction in the domain, it is not global but whatever, it is asymptotically stable, more than enough. Okay? And this is in fact, sounds more general. I didn't have to do all this omega construction and all, right? It gave me asymptotic stability. So basically what it is saying is, if you start anywhere but at this equilibrium, anywhere but this equilibrium, so you can start anywhere arbitrarily close to it but away from that equilibrium, then you will fall here, which is what is the reality also, right? You can verify that very easily in experiments.

Alright? Is that clear? Okay. The spring mass damper example is also very similar and much easier. This is a spring mass damper is just a linear version of the pendulum, linearized pendulum if you may. This is what the dynamics looks like. x1 dot is x2, x2 dot is minus k1 x1 minus k2 x2.

The constants k1 and k2 depend on spring mass coefficients. Alright? And what do I do? I take V as the energy, right? This is the, what is this term? Potential energy. Potential energy. This is the spring energy.

Energy stored in the spring, this guy. And this is the kinetic energy term. Okay? Spring energy, potential energy, kinetic energy. If you take the derivative and of course you can see that this is all nice and radially unbounded in fact. V is radially unbounded. I hope that's clear to you. This is, yeah, in fact in all of Rn, the domain, you don't even have to worry about the domain here.

Sorry, all of R2. Okay? This is V is valid in all of R2. V dot is negative semi-definite in all of R2. And V is radially unbounded in all of R2. Okay? So, once you have that, it's again the same kind of S construction.

Right? Because, sorry, again E has been used. It doesn't matter. So, what is the set E now? The set E is just V dot equal to zero. It is the same. Amazing, no? It comes out to be exactly the same.

You just need x1 comma zero. All sets of the form x1 comma zero. Okay? Now I know zero has to remain. You have to, the second variable of x2 has to remain at zero for all time. Therefore x2 dot has to remain at zero for all time.

Which means that minus k1 x2 minus k2 x2 has to be zero for all time. Same logic. But x2 is already zero. So, x1 also has to be zero for all time. So, what is the largest invariant set? In E, x2 was already zero.

Now I also want x1 to be zero. So zero, zero is the only invariant set inside this. Okay? From every other point you will move. Okay? So, they are not invariant sets. Okay? You cannot find any other invariant set. So, this is the largest invariant set inside E.

Okay? So this is in fact your M set if you know. Yeah? And so any trajectory which starts anywhere in fact will converge to the zero, zero equilibrium. Okay? Again it is a linear system. You could have very well computed the eigenvalues and obviously found out that they were negative.

And you would have concluded exponential stability. This is just giving you an alternate way. But this is just a way of using this theorem. That is all. I mean you have already seen that I can use it for non-linear systems also. In fact you can use it. LaSalle invariance like I said is a method of choice for typically for geometers because like I said because they like to use the energy as the Lyapunov candidate.

And whenever you use the energy as a Lyapunov candidate for these conservative type systems that is with no external forces, invariably your V dot will be zero. Right? Because energy is conserved. Right? It will not be less than equal to anything or will be exactly zero. And from there if you want to conclude anything you have to use LaSalle invariance.

Okay? You have no choice but to use LaSalle invariance. Okay? So LaSalle invariance is a pretty very very strong method. Yeah? I am not sure if we will have time but there is also what adaptive control folks use is the notion of Barbalat's lemma. It is a different way of doing this Barbeshin-Krasov-LaSalle. Okay? Which is more I would say you might find it easier to use and it can be used for time varying systems also.

Okay? Here you cannot. Yeah, you cannot use these for time varying systems. These results in this form. But Barbalat's lemma also cannot be used for multiple equilibria case. So the general LaSalle invariance principle is a very powerful result. Yeah? It is a very very strong result.

You cannot get multiple equilibria type results from any typically any other method. Okay? Not easy. Not easy. You have Poincare-Benediction theorem and all that but not easy. They have all sorts of interesting assumptions which you may not satisfy. Yeah? So LaSalle invariance is actually a very very powerful tool that way. Yeah?