Nonlinear Control Design

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Week 5 : Lecture 25 : La Salle's Invariance Principle: Part 3

So, welcome control to control of non-linear dynamical systems. I think we started a rather interesting lecture last time, maybe a little bit complicated, but we started discussing the Lassalle invariance principle. This is for scenarios like I said, where you have several asymptotically stable systems with a you know nice enough Lyapunov candidate, right. But when you take the V dot that the directional derivative, it turns out that it is only negative semi definite, yeah. And this is a sort of an issue, yeah. And because this only by your typical Lyapunov theorem, this only gives you stability, right.

This is not enough obviously, yeah. You know that in most cases you know that these systems are asymptotically stable just by looking at their behaviour in real life, yeah. Just like this pendulum, simple pendulum, ok. Then of the second motivation was systems with limit cycle behaviour like the Van der Poel oscillator.

I mean you have the linear oscillator of course, then Van der Poel oscillator is like a non-linear oscillator. So, they have this limit cycle type of behaviour which is also something you want to sort of capture or encapsulate in your stability or whatever notion you want to call it, yeah, ok, good. So, in order to talk about the LaSalle invariance, we defined few things that is the invariant set, limit points and limit sets. I hope all of you are clear on these definitions. If you still have some confusion, talk to me now or later, anything is ok, ok.

And then we went on to state the LaSalle invariance principle, alright. So, I will restate it, not state it but I will sort of explain what is going on. The LaSalle invariance principle basically constructs a bunch of sets, ok. So, you basically start with the domain, yeah and if you remember I told you that this is the BR type of a set that you were working with last time where everything holds, all the derivatives are negative and all the nice things happen, yeah. So, inside that set is where all the complications begin in LaSalle invariance, alright.

You have to construct a omega which is a invariant and compact set, ok. It has two properties, invariance and compactness. Compactness was just closed and boundedness in wheels. So, you need a closed and bounded set which is invariant, ok, which essentially means that any trajectory that starts inside this set remains inside this set and because it is a closed and bounded set, it means that you have all your trajectories will remain bounded, yeah. If you start inside the omega set, your trajectories remain bounded inside the omega set because it is not like some kind of elongated cylinder or some funny set like that, no.

It is a closed and compact set, yeah, sorry, closed and bounded set, ok. So, it cannot just escape anywhere. So, already by constructing such an omega, we have in a sense said something very nice about the system, so remember that, yeah. So, although we are not assuming Lyapunov candidates here, we are still making some assumptions, ok. And then in this set, we require V to be positive semi-definite and V dot to be negative semi-definite, ok.

In this set, we do not care what happens beyond it, yeah, we are saying we are restricting our entire analysis to this invariant set. If you cannot for your system find such an invariant set, you cannot apply LaSalle invariance, yeah. So, be very careful, I am not just constructing these sets for the fancy of it, ok. If you cannot construct this omega set, you cannot apply LaSalle invariance, ok. And so within this set, you have these two nice properties holding, ok.

And once you have this set omega, the set E is then constructed by taking the set V dot equal to 0, ok, set of the states where V dot is exactly equal to 0. Obviously, we have said that V dot is less than equal to 0, so equal to 0 is also part of omega set, right. Therefore, the set E is completely inside the omega set, that is evident, ok. And once we have the set E, we construct what is called a largest invariant set inside E. This is the set M and the claim of the theorem or the principle is that if you start your initial conditions inside the omega set, you are guaranteed to converge to the set M, ok, which is again a positive limit set and it is the largest invariant set, ok.

So in this case, we did not say anything about the compactness of M or for that matter, we did not say anything about the compactness of E either. So this is something you need to ask yourself. I mean do you think E is a compact set or can you say anything about the compactness or closed and boundedness of the set E? Ok, let me start simpler. Is the set E bounded? Yes, I mean it is even evident by the picture but you should not again ever give me proof by picture, yeah, please do not do that. The set E is bounded just because it is contained inside omega and omega is bounded.

So E and M both acquire those properties, both are bounded sets. What about closed? Is E a closed set? Is E a closed set? You have done this argument many times. Yes? How are you going to, so you think it is closed? You are doing proof by picture, aren't you? Because I drew this line, you think inside and outside. Those ideas work if you know the spaces. Here I am not giving you any particular shapes and space.

I mean I am making something but E could be very well something very complex looking. Do you think E will always be just a you know closed curve? Not necessarily, right? You could always play with the system just like I constructed all these funny dynamical systems. I could construct it so that E becomes a disc or something like that. Or maybe you choose the V very badly then also this is possible. So is E closed? We talked about two ways of testing whether a set is closed.

One was this complement thing which you learnt. There was another way, right? Which is what we have been using a lot to test whether its set is open or closed. How do you do that? We have mentioned this a few times. No, we never used this in this class if you remember. I mean fine, I mean maybe once.

The same thing what she said. Contains all its limit points means supremum is also a limit point actually. But that is not the test we never used in this class. We used something else to construct open sets and closed sets specifically in the proof of stability. We kept constructing open sets on this side and that side.

How did we do that? Any other way of testing, any other way of constructing open sets, forget proving open sets. How do I construct, if I give you open set in one space, how do I construct open set in another space? Thank you very much. Same with closed, right? Inverse of closed is closed under continuous function, inverse of open is open under continuous function. Excellent. So, in this case what can I say? What is the set E? Set E is what? It is written here.

V dot of, yes, I am hearing something but not. What is set E? And why is it so complicated? It is defined here. What is set E? Okay, you folks are not used to this notation. Please get used to this. This is not difficult at all.

You will, because this notation will show up in your exams. From this E definition, set E is obvious. What is set E? No, no, no, no. What is, no, nothing complicated at all.

From here I have defined E. Can I write E in shorthand in any smart way? You just said it is a function, continuous function. Is there a function involved here or not? All right, wow, okay. Disappointed to say the least, okay. Why? What is so complicated? Why? What is E? No, I won't go ahead without this sensor. I will be completely silenced then.

What is set E? I have defined it. Can it be written as something else? Any short form? Any shorthand? What is the set E? Yes, it is a function of what? Set E is the image of the function. Wow, okay, no. Unfortunately no. So, is it, can I write it in another way or no? Is this it? Just saying words.

I need the math. I can write one thing in 20 different ways in math and we have done this so many times. Why you guys are like, okay. You use these notations all the time. What are we doing? What did we do here? What is it that we did here? Can you say and somebody tell me what is it that we were doing here? How did I construct this E? There is a different E by the way but still how was this constructed? Why? Forget inverse image of open set. Why did I construct it exactly like this? What was the reason to construct it like this? What was the origin of this idea? I mean, you know, because I am not going to spoon feed you.

Here I am telling you the proof and giving you step by step so it seems easy. If I give you a simple proof in the exam, you will not be able to write one step if you falter like this. Yeah. We wanted a bound of norm x. So what? So we wanted a bound of what? Norm x.

We wanted a bound of norm x, is it? I don't think we wanted a bound. What did we want a bound on? Okay. So, so, so we wanted the V to be upper bounded by alpha epsilon 1 square, okay. V of x to be bounded by alpha epsilon 1 square, okay. Then how did I go from here to here? What is the logic? V upper bounded by alpha epsilon 1 square.

So, why this? And so you are saying the, what is, where does the set E lie by the way? Set E is in which space? It is an Rn, fine. I will tell you it is an Rn. So it is in the x space, space where your states are sitting, okay. So V of x is, I wanted to be less than alpha epsilon 1 square.

So from there I went to this one. I just constructed minus alpha epsilon, no don't do this. When you do this in the exam also you have to be able to construct. It is the same logic that I am asking you. You are saying, you are thinking it is obvious because I am telling you the steps. When I tell you the steps everything seems obvious.

In the exam there are no steps, right. You will not be given steps. You have to start from the beginning. So why did I construct an inverse? Because I knew Vx had to be less than alpha epsilon 1 square, alright. Then x has to be V inverse of whatever, alpha, 0 to alpha epsilon 1 square or minus alpha epsilon 1 square to epsilon, okay, great. What am I doing different here? Where is this guy? This guy.

Isn't E also a set in the space of x? Okay, excellent. Now can I write E in another short form? He said E is image of a function. It was wrong but not at least it is a direction. Is E the image of a function? Is E the image of a function? Okay, what is it a pre-image of? Thank you. What is E then? V dot inverse what? What? What? V dot inverse of what? I will try.

Thank you. V dot inverse of 0. By the way, is it just V dot inverse of 0 or is it enough? Is this enough? This is not enough, I can tell you. Why? Yeah, these are things you have to be very used to. You don't state it as a prerequisite unfortunately but I should probably. Analysis is a little bit of a prerequisite here.

So x is an omega. x is not an Rn, not the entire space. If x was an Rn, this is okay. But x is not an Rn. So what? How do I fix this in this definition here? Yes, yes, in words, good.

In math. Yeah, thank you very much. Intersection omega. So, this is precisely what it is. Okay, please be little bit comfortable with this. This will be a bit of trouble for you otherwise.

We will be, you will have to do this. Okay, now I am claiming E is closed. Why? Why is E closed? So these are basically standard set operations. I am not doing any analysis. I am just writing a set in, for example, whatever you are saying in words, these things have to be captured in math. Why do I have to capture it in math? If you do not do that, you cannot claim anything but closed and open.

Just by saying these words, you cannot prove that it is closed or an open set. But as soon as I wrote it like this, I can claim something. Is it closed? Why? Function, yes, yes. Function is continuous, so little bit more.

Little bit more. Sure, function is continuous. V dot is continuous. Okay, because I said V is C1 which means it is a continuously differentiable function. So the derivative is continuous.

So V dot is continuous. Great. Now what? Omega is closed. Omega is closed. Okay, alright, sure. But what about this? Is this a closed set? Why? Inverse image of a closed set.

Because single, I said this last time. In fact, write these few things down. Write these down in your notebooks. What is it? Inverse image of closed set is closed under continuous function. Inverse image of open set is open under continuous function.

Okay. Closed set contains its supremum and all limit points. Okay. Then finite element, finite element sets, that is, finite sets with finite elements are always closed. These are some key facts.

Write this, memorize this. Memorize this. No need to prove or anything. Memorize it. Nobody is asking to prove it.

It is not an analysis course. Inverse of open, open. Inverse of closed, closed under continuous function. So, finite sets always closed, closed sets always contain their supremum, infimum and all limit points. Supremum, infimum are basically limit points.

Okay. Supremum and infimum are also limit points. Okay. Okay. Good. So, this is E is closed.

Okay. I found that E is closed. Can I say something about M? M, M. It will lead you to more facts that you have to copy down. What about M? M is bounded. M is already set. M inherits the property of omega.

Is it closed? See, M is already invariant. Right? We already said that. M, that is what we defined it as. It is the largest invariant.

So, let us forget about that. It is bounded. Closed is the only property that is left. Is M a closed set? So, M is a positive limit set. I already mentioned it here. Okay. Are limit sets closed sets? Anybody know this? Limit sets are what? I defined it.

Right? It is a set of limit points. So, it is the points to which the sequences converge. Now, what is a closed set? A closed set is a set which contains all its limit points.

I just said that. Okay. It is one of the points. I hope you have noted it down. Okay. Now, what I am asking is, is the limit set itself a closed set? Which means, which means, does the limit set contain all its limit points? Almost sounds like I am doing a rap song, but I am not. Does the limit set contain all its limit points? Okay.

Even if you do not know, the answer is yes. Okay. Because a limit set will have no limit points other than the elements of the set itself.

Okay. Think of the circle here. This is a limit set. We discussed this. Right? In fact, this is a two simple element set. Wherever you start, it will just making the circle. So, forever it is there.

Okay. So, if you see, there are no other limit points other than this. Okay. Remember, the limit points depend on the initial condition. So, do not get confused by the other circles.

So, all of this is corresponding to an initial condition. Okay. So, I apologize. So, this, the limit set, the limit points of the limit set are the limit set itself.

Okay. Note this down. Limit points of a limit set are the same set. Okay. So, if you call omega or you can just write in short hand omega if you can, if you remember from our notes. Limit points of omega are the set omega, cannot be beyond omega. Okay. Which means omega is a, sorry not omega, M, M, M, M, sorry M.

M was the limit set, not omega, M. So, limit points of M or any limit set are the same set itself, cannot be beyond that set. It is, you can check with any sequence. If you are, even if you are not able to prove it, you can check any sequence. This half, one, half, one sequence, the limit set is one and a half, one comma half.

Right. It can have no other limit points because anyway it is two discrete points. So, it is a closed set. Therefore, M is a closed set.

Limit set is a closed set. Okay. These are facts that I want you to memorize. We will not have the luxury to prove it right now. Okay. Good. So, we have a bunch of very nice sets.

I hope you are now convinced that all our sets are super nice. Except for the D domain set, inside that we had omega which we constructed, which nobody gave it, gave to us, which was compact and invariant. Inside that we found in set E which is also compact and invariant now. Because it has closed bounded, sorry, it is only compact. The set E is only compact, not invariant.

And then inside that we found the largest invariant set. Therefore, M is also compact and invariant. So, M has exactly the same properties as omega.

Okay. So, you start from a similar type of set and you end in a similar type of set. Okay. So, this invariance is, that is why geometres love Lassalle invariance principle because it is very mathematical. Okay. But this is also very useful.

Okay. So, just keep this in mind that all these sets have very very nice properties. And the ability to apply the Lassalle invariance relies on you having this omega set. If you do not, cannot apply.

Okay. We will look at the proof. But before that, I remember we were doing the example. So, we will restart the example.

Okay. This was the pendulum. Correct. And what did we do? We took just the energy as the V function. Okay. We do not care, we only need semidefiniteness. And it is, okay, it is, in fact, if you take any x1, x2, it is semi, it is not going to be negative because of 1 minus cos x1 is lower bounded at 0, same this way. So, it is semi-definite.

If you want positive definiteness, then you have to fix x1 to a small range minus pi to pi. But we do not care to have positive definiteness. Okay. We are, semi-definiteness is enough. So, we choose a larger range of x1. Why? Why did we choose a larger range of x1? Do you remember? For the pendulum? To have more than one equilibrium.

Because we want to show the power of LaSalle invariance. So, we were capturing this equilibrium and this equilibrium also by x1 minus 2 pi to 2 pi. But x2 was free to be within R and this was our domain D.

Okay. Now, to construct omega, we did some interesting manipulation. Okay. How did we construct omega? We constructed omega using V itself. Okay. So, we did somehow the reverse thing. First we chose a V and used that to construct omega.

How did we do that? We said that, so first we did this V dot. We saw that it is less than equal to zero. Okay. Wherever we take, it doesn't matter. It is definitely negative semi-definite. The domain of x1, x2 everywhere in D in fact.

Anywhere in D, this is negative semi-definite. So, I am not concerned about the set omega anymore. So, this V will help me construct the omega. So, what do I know? V is non-increasing.

Okay. So, if I start the V at some value, it will stay below that value. Okay. So, I am going to construct the set omega such that V remains below this value.

Okay. And how did we do that? I think we did, we wrote it. We write it. I think I erased it. If you remember, we construct the set omega as, I wrote this by the way, x1, x2 in D such that V less than equal to some constant C. Okay. And this is the same as or this is same as saying V t0 less than equal to this constant.

I am more than okay because V dot is negative semi-definite in all of t. Okay. And to do this, I just wrote this guy. I want this quantity to be less than equal to C. Okay. So, how do I do that? I take the smallest value of this which is 0.

Right. And then I have x2 is bounded by square root 2C. Okay. This we calculated last time.

So, therefore, I get my omega as this set. Okay. I get my omega as this set. Okay. Just note that it is written here. This is from how I get this. The only thing is I will not make them open ended.

I will make this closed. Okay. I have made this closed just to ensure omega is a closed set. If I take the open bracket, then it is an open set.

That I do not want. I want omega to be closed. So, this is what I take my omega as. Okay. This is fine. I mean, in fact, I can take my domain also. Domain has no restriction.

It can be open or closed. Okay. So, I take the domain also with the closed bracket, not the open bracket.

Okay. So, so this is my omega. I have a nice omega now. Okay. Now, I am ready to apply the LaSalle invariance. Forget this. Yeah. So, how do I apply LaSalle invariance? So yeah. So I construct the E set.

What is the E set? The set where V dot is exactly zero. Right? So that is just requires me to have the X2 term to be zero. Okay.

I do not care about the X1 term. Okay. So, this is the set E. Again, I will make this the closed bracket, not the open bracket. Okay.

Yes. Why are we taking the maximum value? We are taking the minimum value. Yeah. Because it is just an inequality requirement. No. Look at this. I want VX less than equal to C.

We have VX less than equal to C. So, this is I want X2 square by 2 plus K1 minus cosine X1 less than equal to C. Okay. You can look at it in a couple of ways. First is I am not interested in bounding of X1 because X1 is already bounded the way I want.

Okay. So, this is only going to make a positive contribution. Right. This is only going to make the left side larger. So, if or else you can write if you may, I will write this. Do you believe me? Huh? So, effectively I made this zero only. See, if I took the maximum, what will I get? I will get this as less than equal to X2 square by 2 plus 2K.

But that is not helpful for me. This is also an upper bound. This is also an upper bound. I cannot compare the two. How do? If the VX is less than equal to C, then max value is going to be less than equal to C.

No. I do not require that. I do not require that. I only need VX to be less than equal to C. Why the maximum value? Okay. It may not reach the maximum value. Understand? Yeah. So, so this will not be, whenever you are comparing inequalities, this is pretty standard. You have to be able to write it like this.

Once I write it like this, I can compare this and this because it is a natural inequality going from left to right. But if I write it like this, I cannot compare this. There is no guarantee which one is larger. This is standard in inequality. I know you are thinking max of V has to be, but no, it does not have to be because the max of V may never be achieved.

But we are taking the point of X and Y. Yeah. We are taking, but it may not hit all the sides. We do not care. We only need this much. Because if you do this, you are significantly restricting X2, significantly, which we do not need, which we do not need, which we do not need. We come up with the best estimate or largest possible estimate of X2. Okay? Alright. Thank you. Thank you.