Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 4 : Lecture 21 : Proofs of Lyapunov Stability Theorems- Part 3

Stage 2, now we go to the general class K function, nothing much changes you already have this stuff, so this is going to help you anyway, but now all this alpha epsilon square cannot happen, I hope that is evident because alpha epsilon 1 square came because I assumed alpha norm x square as my class K function that cannot happen. So I restate everything, I have V t x greater than equal to phi norm of x, alright for all t greater than equal to 0 for all x in B r and I also have V dot to be semi-definite again under these assumptions. So I write the same kind of statement, this is the same kind of statement that I wrote that there is a lower bound phi norm of x which is this, basically I am rewriting this here and then I have an upper bound this guy, this is just coming from my semi-definiteness condition, this guy is just from the semi-definiteness condition. Now what do I claim or what do I want? Earlier I wanted alpha epsilon 1 square on the right hand side, now I say I just have on the right hand side less than phi epsilon 1, yeah I just use this function itself, so I have phi epsilon 1. Now remember these functions phi have beautiful properties, they are 0 at 0, they are continuous, they are strictly, they are monotonic. Therefore if you have phi norm of x to be less than phi epsilon 1 which is what you do, then norm x has to be less than epsilon 1, I hope this is sort of evident to you, 0 at 0 monotone increasing function.

So if the function value is less than phi epsilon 1 then the argument has to be less than epsilon 1, there is no two ways about it, so this is rather neat sort of a result, okay. And this is also, in this case there is invertibility in play, yeah because the both sides are real numbers, argument of phi is a real number, what it outputs is a real number, okay. And phi invertibility is guaranteed by what? Monotonicity, the monotone function, invertibility is guaranteed, done. So if phi x is less than phi epsilon 1, norm of x is less than epsilon 1, okay.

So if I can prove this that this happens just like before instead of alpha epsilon 1 square I just have phi epsilon 1, then I am done, okay. So not too different. So what do I want? For this to happen, so I just have to write it in slightly more complicated language, that's all. Here I had minus alpha epsilon square epsilon 1 square, alpha epsilon square, here I have to write it as this way. This is what we need for this to happen, yeah.

This condition is rewritten as this, I hope you understand, just by taking the, just this condition is rewritten as this way, okay. Just by taking phi inverse on the left hand side. So it is just saying that because I have to write everything as open sets in some sense, like open sets, I mean these are relative open sets, so things seem complicated, but this is what I want. I want phi inverse of 0 to be t0 x0 to lie within 0 epsilon 1, okay, alright. So here this notion of open, inverse open and all seems a bit murky here, but don't worry it is not murky, yeah.

What happens is we are talking about relative open sets, so don't, I am not going to explain it, don't worry about it, that open issue is not giving, I mean taking a beating here, alright. So we know that phi is increasing and continuous, right. So I have drawn a sort of a picture here, right. So I know that if you look at this picture here, vt0 x is continuous because I am again fixing the t0, right. So it is a function of x, therefore this notation.

So vt0 x is continuous, it is 0 at 0 here and phi epsilon 1 is greater than 0 here, right. Therefore there has to exist some norm x bound, right, some x bound such that vt0 x lies within this range, okay, just by continuity of v, alright. Just by continuity of v I can get this range, okay, alright. Now, yeah, so that is what I say here, I say here in this more mathematical language that we choose delta such that sup of norm x less than delta vt0 x is less than equal to this guy, exactly the same statement as before. Here it was alpha epsilon 1 square, now I have just written the phi epsilon 1, nothing has changed, exactly the same argument, okay, okay.

So, alright, so I will go to the aside later on, but if this happens, so if x0 is less than delta then I do have from this condition that vt0 x0 is less than phi epsilon 1 and if this happens, right, I know that vtx is less than equal to vt0 x0, alright, because delta is less than equal to R and norm x0 is less than equal to delta, okay. So which means that norm x0 is also less than equal to R, so I am in the good place where all my negative semi-definiteness etc. hold, okay. So if delta is less than equal to R that is what I have assumed, then norm x0 is also less than equal to R, okay, which means I am in a good place. So vtx is less than equal to vt0 x0, okay, so this also holds.

And once this holds, you have of course that phi norm, the first statement here, I have just repeated that here, phi norm of x less than equal to vtx less than equal to vt0 x0 less than phi epsilon 1. The only purpose of this statement was to sort of tell you that your initial condition is within the R ball, okay. And if the initial condition is within the R ball, you have some space to go, again the R ball is also an open set. So if your initial condition is within a delta ball, within the R ball, there is some more space to go. So you are within the R ball, your analysis is going on within the R ball, your trajectories are still within the R ball.

So if you start within the delta ball, then this negative semi-definiteness will hold because your trajectories are within the R ball, if you started in the delta ball. So therefore this negative semi-definiteness holds and therefore you just add these two pieces from the beginning, that's all. These things are of course also holding because you are in the R ball. So once you have this, you have norm x is less than epsilon 1 which is less than epsilon and again less than R, okay, so continues to hold. So the only thing that I did not prove is this guy which I am saying not exactly, I did not exactly prove but I sort of indicated to you that this is again going to happen by continuity, okay.

Because continuity will give me some bound on x and once I get some bound on x, I will get a bound on norm x. It can be conservative or whatever, it doesn't matter, I will get a bound. The aside that I want to sort of say here is that is this particular sequence of things, okay. So delta has to be upper bounded by epsilon 1 which is upper bounded by R. This is evident by the choice of epsilon 1 itself and epsilon 1 is defined so that this happens.

So this is not complicated of course but I am claiming that this has to be the case, okay. This has, remember when we defined stability, we already said delta is less than epsilon or less than equal to epsilon but that was in the definition of stability. Here we are trying to prove stability. So this delta that we are getting is not from the stability definition. This we are getting from here, okay.

So it is important for us to sort of prove that delta is going to be less than R because if delta is not less than R, then this cannot be claimed. I hope this is clear to you. If delta is not less than R, this is not true anymore because your initial condition may already be outside the R ball. Then negative semi-definiteness does not hold. So that is not somehow evident here.

Just by looking at this that whatever delta you get, will it be less than R or not is not evident just by looking at this, okay. So that is what I have just tried to prove very quickly, nothing too complicated, yeah. So what I am saying is let us assume for contradiction that delta is greater than epsilon 1, okay. And if delta is greater than epsilon 1, then I know by my positive definiteness that this happens. This is just the positive definiteness statement which means that by monotonicity of phi, I know this is true.

Why? This is true when this happens. I am assuming that delta is greater than epsilon 1. So there exists some norm of x between epsilon 1 and delta, correct. Delta is strictly greater than epsilon 1. So there is some value in between.

So I can choose a norm of x is in between that value, yeah, because delta is greater than epsilon 1. So there exists some norm of x in between. Norm of x is just a number, right, just a number, okay. So there exists some number in between. Now if norm of x is in between this epsilon 1 and delta, then if norm of x is greater than epsilon 1, I know that phi norm of x is greater than phi epsilon 1, right, by monotonicity of the phi.

So what have I just proved? I have proved that Vt0x is greater than equal to phi epsilon 1 for some norm of x, okay. And this norm of x is still within the initial condition bound because the initial condition bound was less than delta, right. So this norm of x is still satisfying this initial condition bound. And within this bound, I have now proved that Vt0 is greater than phi epsilon 1, okay. But this is a contradiction, right.

This is a contradiction. I chose my delta such that this upper bound holds, okay. So there is a problem with my assumption. There is a problem with my assumption, okay. So this assumption is invalid. This is one of the ways of proving results in mathematics by contradiction, alright.

So that is what we have done. So just I have assumed that there is a contradiction that is delta is actually greater than epsilon 1 and then I show that something goes wrong here, alright, which it cannot. I am not allowing it to. My delta is chosen in this way, okay. So it is important for us to sort of ensure that delta is less than R so that this satisfies. And if this satisfies, then I have these two additional things and I am done, alright.

I am done with the proof. So in the general phi case also, I can do this. The only thing is it is not very evident that this e and so on and so forth, how the picture looks and how the open sets look. But it still, the proof goes along in exactly the same lines. You basically have this sort of a, see, in this, when there was alpha, I had a 1 by alpha here, right, instead of phi inverse. I just had a 1 by alpha here, alright.

Here just I have a phi inverse, okay. So basically I had something, so here I had a alpha epsilon 1 square, right. So that was the idea, not too complicated here. Now I know I have written it in this form. I am wondering if I can write it somehow in this form also to construct the set E.

Can anybody of you suggest how this set E will look in this case? So here the equivalent was just alpha epsilon 1 square here, right. That was the only difference here. There was an alpha epsilon 1 square here. So how do I define the set E? Can anybody tell me? It is something V t 0 inverse of minus phi epsilon 1 to phi epsilon 1, okay.

That is it. Same deal. I know I wrote it in a different way like this, but you have to worry. This is it. E is just this set, okay. And remember P epsilon 1 is just a number. So therefore this is an open set, right, just by previous notion, right, inverse of open set under continuous function is open.

So this is also an open set and this is an open set means I just have this picture again, some E, yeah. We saw how to get the equation, right. Somehow you have an equation. It could be an ellipsoid. It could be some funny shape, does not matter.

Important thing to remember is that origin is contained in this. Why is origin contained in this? Because origin is in this set and inverse under this function of origin is origin by definition. Therefore origin is, so in fact I can even say something like this. 0 belongs to E, okay.

Origin is contained in this. So origin is in E and E is an open set. Same deal. Make these things. Then I get a delta. So it is evident now that I got the delta, right.

I mean it is not evident maybe from this whatever but this expression but it is evident from this picture. Not constructive, do not expect constructive things in general non-linear function cases but it is a delta, choice of delta, fair enough choice of delta, okay. Alright, questions? Comments? Is this too complicated? This is more or less, I mean well the LaSalle invariance proof is a little bit more complicated, little bit more. I mean that is, it is the geometry guys.

So that stuff is always more complicated. But this is fairly straightforward actually. Yeah, I am trying to wonder if exponential stability proof goes simpler or does not go simpler. I do not particularly think there will be any, yeah I am not sure there will be any particular advantage there either. Because there you have to use the same order of magnitude idea to get a exponential decay. See here in all these proofs again as is expected until now anyway we have just talked about stability but we have not, there is no rate of convergence even when we look at asymptotic stability next you will see that there is no particular rate of convergence notion as such.

So you cannot expect any rate of convergence idea. Anyway, so the exercise is to complete uniform stability. How do you think you will go about it? So I hope it is evident again, again I hope it is evident that here I am taking a VT0 inverse, here I am taking a VT0 inverse of this guy to find a delta. So the E is somehow I mean dependent on t0 and of course t0 and epsilon because epsilon is right here inside this.

So E is a function of t0 epsilon. So if I want to get rid of this t0 that is what I will need right to prove uniform stability because if not then if E depends on these then delta has to depend on this. The set E is depending on this. So the only way for me to get rid of t0 dependence is to get rid of t0 dependence in E. If E does not depend on t0 or initial time then I have arbitrary choice of delta which is independent of t0. So how do you think I would be able to remove that t0 dependence here? This is fine.

What about this t0 dependence here? How do you think I can remove it to prove uniform stability? How do you prove anything when I ask you to prove something? Anyway you should when you do that hopefully you will be a little bit more comfortable with these. See some of you have already done proofs but mathematical proofs are you understand that they are different require little bit of a different mindset but eventually whenever you prove anything what how does it matter how does it work? I give you some statement generally I am just not talking about this or anything I am just saying there are some statements or assumption and based on those you get a result right. So what are the assumptions in this case for uniform stability? Yes. What is the uniform stability Lyapunov theorem? We use decrescent.

We use decrescent. Alright. So that is what I am assuming should be useful to you because otherwise in stability proof we used all the other features of V and V dot right. The only thing that we did not have and use was decrescence. So obviously I need to use that no otherwise I cannot just prove an additional property for free without assuming something.

So you can think what you can do with decrescence alright. Okay I think that is it we will stop here. Thank you.