Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 4 : Lecture 20 : Proofs of Lyapunov Stability Theorems- Part 2

So, like I said you have to choose this v t0 x0 such that this happens. Alright. Now, like I said I am claiming it is possible to choose a delta such that the supremum of this guy for over norm is less than delta does satisfy this. Okay. So, how do you claim this? So, this requires a little bit of your real analysis type knowledge, but it is not too complicated to grasp. First thing is I want to just use this notation which to describe this function.

So, this is v t0 subscript t0 x is just the function v t0 x. Just notation not a big deal just to make my life easy to write things. So, you know that v t0 0 is basically this guy and that is exactly 0. Yes.

And you also know that v t0 is continuous which means that this function is continuous in x. This is an assumption. In fact, it is C1. You assume that this is C1 function and whenever I say it is a C1 function it is C1 in both arguments technically. So, it has to be C1 in this guy also.

So, definitely continuous in x. Alright. So, the good thing is you have, let's look at this picture. Okay. This picture right here.

You know how this function works. Right. v does not have to be an increasing function or anything. It just has to dominate this class K function alpha norm x square in this case. We have already seen this picture.

Right. On the x axis there is norm x and on the y axis there is value of v. Alright. Now, what do I want? I want this v to lie within this range. Right. I hope this is visible.

So, I want v to lie in this range. Right. So, what range does it give me on norm x? First of all, you have to be relatively convinced that this is a good picture and a representative picture. There is no such thing. I mean, don't even try this.

Don't do proof by picture in any exam. I am using the picture only to support, help you understand. All I am trying to show from this picture is that this red guy is any continuous function. Remember.

Okay. Which means what? Which means that if this is any continuous function and you notice that the value I have taken is alpha epsilon 1 square. Right. So, this is basically you know that x equal to epsilon 1 will satisfy this. Right. At x equal to epsilon 1 this function will have this value.

Okay. So, so the important thing to remember is that this is a continuous function. So, it must take all values within a range. Okay. It must take all values within a range.

Alright. So, it has to also take the value alpha epsilon 1 square. This is basically the intermediate function theorem. Intermediate value theorem. Sorry, not intermediate function theorem. The intermediate value theorem.

Basically, it says that if you have a continuous function which maps a, b to you know p, q then it has to the function has to take all values between p and q. It can't just miss some value simply because it is a continuous function. I hope it is intuitively clear. Of course, there is proofs of it but intuitively also it is evident.

Right. So, so remember that this alpha epsilon 1 square. Okay. Is, is within the range of the function, within the image of the function. So, this value also has to be taken. But in any case we are not even interested in alpha epsilon 1 square.

Right. The important thing is we are interested in something below it. Okay. We are interested in something below it. Okay. So, what am I now going to say? I am only looking for the value of the function to be here.

Yeah. So, although I have written it. So, I am going to define a set. I am going to define a set E which is just this horizontal line by the way. By this. What is this set? It is just taking the range.

This is open set by the way. Minus alpha epsilon 1 square to alpha epsilon 1 square. This is an open set. I hope this is clear to you.

That this is an open set. And I am taking its inverse image under V t 0. Okay. So, this is actually, let me not say this is V t, t x but this is actually V t 0 x. Because we are only interested in doing our analysis for V t 0 x. We don't care about V t x in general.

We are not varying time. We are just fixing time at t 0 and when we are looking at that function. Because I just want to bound V t 0 x. That is my job right now. So, this plot is just for V t 0 x.

Okay. I hope that is simple and clear. That way I can also make this kind of a plot. Because if time is also changing then I can't make this kind of a plot. We already discussed this. So, I am creating this set E as V t 0 inverse of this open set.

Okay. Alright. Now, from your analysis course I hope or whatever analysis class you have seen, you know that if you have a continuous function and you take the inverse of a open set under a continuous function then the inverse is also an open set. So, open set taken the inverse under a continuous function then E is also an open set. This is a fact of analysis. In fact, most of us do the in real analysis most of us do things the other way around. This is actually the definition of open sets in a topological sense.

This is actually the definition of open sets. In analysis we do it differently. We say the definition is something else and then we derive this as a theorem typically. But in reality this is the definition and what you define in analysis courses as open sets is a theorem.

Okay. So, just flipped version, but the basic point is both are equivalent. Okay. So, if you take an open set take its inverse under any continuous function it has to be an open set.

Yeah. Disaster. Alright. Okay. Alright. Great. But for us the thing to remember pictorially is very simple. This open set E is just this guy. Now, I know you are mighty confused because I used minus alpha epsilon 1 and plus alpha epsilon 1.

Here and here. But the point is my intersection is still this set. Yeah. I hope you understand because there is no image here on this side for V. Okay. So, the inverse of this set under V is still does this set only.

Because there is nothing in the bottom half at all. So, V is defined in that way. Okay. I have just why have I used it in this way because I am just trying to make it look simpler right in the sense that it is actually looking like an open set. In fact, we are talking about relative open sets, but I am not going to even discuss that.

So, the idea is the inverse under V t 0 of this set is only this set. Okay. It is exactly this set. It is basically the set norm x less than delta.

Norm x less than delta. Why? Because I have open here, open here.

So, I have open here. Okay. Not open here. No. That's wrong. It's open here, open here. This gives me a set which is open here.

Still contains the origin. Okay. Still contains the origin. Remember that. Yeah. Because origin is contained in this set.

In the vertical set. Yeah. Just in this picture. The vertical set contains the origin. Therefore, the horizontal set also contains the origin.

Okay. Because 0 maps to 0. V t 0 0 is 0. So, origin is contained in this set. That should not be in any contention. Okay. So, that's what I am saying.

This set E is open and it contains the origin. Okay. Okay. And this set is just basically this guy. And this quantity is what gives me the delta. This length is what gives me the delta.

So, basically such a delta has to exist. Just by continuity. Just by continuity of the function V t 0.

Okay. Such a delta has to exist. Alright. So, this is the cool thing about lot of mathematical results. It is not constructive. I am not actually giving you a value of delta. And you can't, I can't. For me to give you an actual value of delta, I will need the dynamics, I will need the V.

I will need so many things. I can't give you the actual value of delta. But I know that this delta exists. Just by continuity of this V t 0. So, you see all the ingredients of our Lyapunov theorems, how do they get used? Rather nice.

Okay. Great. So good. So, you know that 0 is, E is open and 0 is contained in E. Excellent. I have drawn another picture. So, whenever I teach this analysis type course, this structures, mathematical structures course, I make a lot of pictures.

They are very helpful in following things. Yeah. So, if you have this set E, I am saying origin is contained in this set E. And I am saying E is an open set. Okay. Now, I know that this may not directly give you a delta, but this picture seems to indicate it gives a delta.

In matrix spaces it is very easy. Yes, this gives a delta. But even if you are not convinced I still have an exact delta with norm x less than delta, it is not difficult to follow. Because see, why I am making this another picture is just to generalize it to functions of the form phi norm of x. Right now you had what? I had taken a special case. Right? That phi norm of x is actually alpha norm x square.

It is a special case. But if it is not this special case, then your E might be in some funny shape. It may not be in a ball shape. Okay. And if E is not in a ball shape, no problem.

Because origin is inside this open set. Okay. And the definition of an open set is what? How do you define an open set? A set is open. No, that is just something you have depicted in your mind. You take any element of that set, there will also be a ball around that set that will be in that set.

Absolutely. Okay. Basically, if you take any element of the set, there exists a ball around that point which is also inside the set. So since origin is inside the set, there exists a ball which is also inside the set.

Okay. And this is my delta ball. Done. I got my delta ball. So even if I take the general case, here I took the special case, so I immediately got a ball, norm x less than delta. But even if I take the general case, where my E no longer looks like a ball, it looks like some kind of an ellipsoid. Typically you will have ellipsoid or whatever.

You may have some crazy shape. Nobody cares. It could be this. I don't care what it is. Yeah. But the point is because it's an open set and the origin is inside that set, there exists a delta ball which is contained in the set.

Okay. So that's it. Whenever x0 is less than delta, whenever I am inside this ball, I am inside the E set. And if I am inside the E set, my vt0 can map me only inside this value. Right. Because if I take this to the other side, vt0 of E is, has to lie within this range.

Okay. So I am done. I have just proven that if I take norm x0 less than delta, vt0 is always less than alpha epsilon 1 square.

Okay. Okay. This is a very analysis-based proof. So just think that you followed. You think you followed great. If not, ask me. Okay. Very simple. We constructed an open set by taking inverse and this is why you remember I told you inverse of the V functions are involved.

Okay. Okay. Alright. Okay. So again, so there is a lot of subtle point that I don't talk about but I hope you understand this first. Just take the E set which is the inverse of this open set. So I constructed an E set. Now within the E set, I know I can get a delta ball. So all my initial conditions starting inside that delta ball are inside the E set and because they are inside the E set, they can never map me out of alpha epsilon 1 square.

So the delta is obviously a conservative ball. Remember. This could be very small but who cares. We need to just satisfy the stability definition. Nobody asked us what is the delta. So basically I have just proved that V t0 x0 will be within alpha epsilon 1 square if you start in this place.

Excellent. We are done. What are the subtle points here? Has anybody got any subtle points? Remember we said that positive definiteness is required if you have, if you don't have radial unboundedness, you can't do global things. Where does that play a role in here? This is the stability we said. It doesn't play any role. But anyway, suppose I gave you a radially unbounded V or a globally positive definite V.

Forget radial unbounded. Suppose I gave you a globally positive definite V, that is this Br was not there, then what would change in this proof? This is a good way to learn proofs by the way. To see where your assumptions went, if I change something, tweak something a little bit, what happens? So if I gave you a positive definite V, that is globally positive definite, that is it is positive definite for all r. Similarly V dot is negative definite for all, negative semi definite for all r. What will change in the proof? What will change in the proof? No need for epsilon 1.

I don't need to define any epsilon 1. That is the change. So when you give me any epsilon, there is no longer an r or anything. There is just epsilon, directly work with epsilon. That is the only difference here. But the stability result doesn't change at all. Remember to obtain stability, positive definiteness locally is enough, as long as you are working in the local range, in the r range.

Beyond that, that is a problem. How do you think we, do you think we are guaranteeing somehow that we are within the r ball? Because I said, right, trajectories have to assume they are within r ball, otherwise I have a problem, nothing works, right, because I don't have positive definiteness, I don't have negative semi definite. My proof is irrelevant beyond the r ball, right, because I don't even have positive definiteness. So all the arguments I used are ridiculous. How or if, how or whether am I enforcing or guaranteeing staying in the r ball? Am I doing that in any case? Or am I just assuming it? See, God's grace or whatever, good fortune.

Initial condition is within r ball. Where do I say that initial condition is within r ball? We are starting at t0, not from t0, at time t0. No, I didn't say anything about r, I only said delta, right. Where is r coming up? r is coming in epsilon 1, right. Now, am I somehow saying that, you know, that I will, am I somehow also proving without actually talking about it that I am going to stay within the r ball? Why is this complicated? What have I proved? In the end what have I proved? Say that again. That I proved in the next screen, correct, here, that is that vt0 and x0 is less than alpha epsilon 1 square, correct.

But I used that to prove something else, no? That was not the key thing that I wanted to prove. What did I end up proving here? Say that again. Now what is the upper bound? Epsilon 1, not square, right. I mean, I can get rid of the square here, right. But this, but, but norm x less than epsilon 1 means that norm x is less than equal to epsilon, right.

But it means something else also, right. What else? Which ball? r ball, right. Because norm x less than equal to epsilon 1 means that it is less than equal to epsilon, but it is also less than equal to r, right. Because epsilon 1 is the min, smaller of the two. So if norm x is less than equal to epsilon 1, it is definitely less than equal to both of them, right. So I have just, I have actually proved that by this analysis that norm x is going to remain within r.

So I am going to stay within the r ball just by virtue of this. Why do I get this for free, seemingly for free? Without actually aiming to prove this is because I have semi-definite, negative semi-definite. So wherever I start, I do not necessarily decay, but I don't necessarily explore. Not necessarily, I definitely don't explore, okay. So that's what, that's what I get that my, if I start in the appropriate delta ball, I am not just guaranteeing that I will be within the epsilon ball, I am also guaranteeing I will be within the r ball. That's the whole point of choosing epsilon 1, that you never escape the r ball also because you are not allowed to, okay.

So just by virtue of this proof, I am guaranteeing both, not just one of them, okay. I am guaranteeing both properties. Thank you.