Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 4 : Lecture 19 : Proofs of Lyapunov Stability Theorems- Part 1

So, welcome to SC 602, our course on Control of Non-linear Dynamical Systems. Yeah, we are well into the, you know, midst of things. We have already started doing the Lyapunov theorem which is the key results in analysis of stability of systems. Okay. So, to recap, we were doing these Lyapunov stability theorems. Alright.

So, we have already defined what they are and what the stability notions are and now we have gone ahead and started to already discuss the Lyapunov stability theorems. Yeah, we started with the more basic version, stability, uniform stability and then you have asymptotic stability, uniform asymptotic stability, global uniform asymptotic stability, then the two notions of local and global exponential stability. Okay. So, we saw that once we had the setup of positive definite functions, radial unbounded functions, the decrescent functions, the statement of the Lyapunov theorems was really easy.

Okay. Really easy. Not difficult to verify either. Yeah. Once you understand what is positive definite function and I really hope you do.

Yeah. I know there were some mistakes last time when I asked a few questions. So, remember these few points. If some state do not appear in a function, it cannot be definite. Yeah.

If you have things like x1 plus x2 whole square cannot be definite. Yeah. So, we have to, so these things somehow should not be material that you need to look at your nodes to sort of remember. Okay. So, if this is something you need to remember, it should be at the back of your mind that these easy tests, right, that if the function can be zero anywhere other than the origin, other than the zero state, then there is a problem.

Okay. These are problems. Okay. Similarly, we also know that some keywords are associated with some results, right. We know that positive definiteness is connected to stability and asymptotic stability. Similarly, negative definiteness gives you asymptotic stability.

Similarly, decrescence is connected to uniform properties, uniformity and finally radial unboundedness is connected to global properties. So, although I was very clear, I did not mention all properties, okay, separately, yeah, but it, this is something that should be relatively clear to you that I can add and subtract one word and I will get the requisite property. Okay. So, like I said, we defined uniform asymptotic stability locally and we defined uniform asymptotic stability globally, but we did not talk about global asymptotic stability, right. So, if I just drop the decrescence here, I will immediately get global asymptotic stability which is not necessarily uniform, okay.

And remember that for nonlinear systems, these are, none of these properties are easy to obtain, yeah, it is not like something is free. The other thing that I have also mentioned, so these are all points to remember, yeah, some of these points that I mentioned are key points, yeah, the easy tests, the word associations and finally that, you know, if you, if you, you know, add and subtract some words, you get some properties and so on, that's evident. And finally also that when you go to exponential properties, they are naturally uniform and also if your dynamical system, the right hand side does not explicitly contain time, then uniformity is free, okay. And so, some of these are things that you just have to keep at the back of your mind. These are not things that I would like you to refer to your notes for this, yeah.

It's almost something you memorize, yeah, just memorize these, yeah. Similarly for linear systems, asymptotic stability, exponential stability are the same, like linear time invariance system, but I can actually solve them and you have seen, we have done the examples that it essentially gives you, you know, your exponential, the rates of convergence are always exponential, okay, alright, great. We also did a few examples where you could, you saw that I sort of played around with some systems just to get some properties and so on. We also looked at finally the pendulum system which again I have modified it because I wanted to make my analysis simpler. We will look at the more or the actual pendulum without this guy later on when we look at LaSalle invariance because we already saw last time that without this term, right, without this term what goes away? Without this term, this term goes away, right.

And if this goes away, V dot is only negative semi-definite, yeah. And that's again a problem because all we get is stability, whereas in reality we know that damped pendulum is just going to stop, okay, it's just going to stop, it's not going to, it's not going to just stay bounded and things like that. Here I am moving my hand, you can see, that's why it's staying bounded, but actually no, if I stop my hand, it's going to stop, okay. The damped pendulum is always going to stop, okay. So that's the reality, but we are not able to prove it with this function which is the energy, okay, which seems like the most natural choice for the Lyapunov candidate in this case, yeah.

So for such cases also we have results like the LaSalle invariance and the Babalach's lemma, so we will discuss them subsequently. So when we do that, then I will drop this term and we will do the analysis, okay, alright, great. Then finally there is anyway some example, but anyway, this is not such a complicated example anyway. Here you have this second order system, alright. This is a worked out example in the notes itself, so it's, that's why it's relatively easy to follow.

Here you have a second order system, you can see that without these terms, if you get rid of these terms, what is it? What is this system? Harmonic oscillator. Harmonic oscillator, just a harmonic and you know that it is just going to go in circle, circle, circle, so just stable and it's not going to be asymptotically doing anything. So let's see what happens because of addition of these terms, alright. Again the authors who came up with this example are also doing nifty things like I am, right. They are just playing around with terms so that they can get good results.

I don't think this is any real system, alright, but anyway, so it's not so uncommon is what I am trying to tell you, alright. So if I take this Lyapunov function, very straightforward, right, this is what I took for the harmonic oscillator also, right and you know that this is radially unbounded and all that. So if I take the derivative, right, x1, x1 dot and x2, x2 dot, for the harmonic oscillator I would have got 0, right. We just did that example, but here because of these additional terms, I will get C x1 square x1 square plus x2 square and C x2 square times x1 square plus x2 square. So if you actually combine them, you will get C x1 square plus x2 square whole square, alright.

Now it should be obvious to you that depending on the sign of C, I will get some definiteness, okay. If C is negative, I will get negative definiteness. I will get basically, v dot is negative definite. I hope you believe that. This is not like x1 plus x2 whole square.

This is x1 square plus x2 square whole square. So this is actually a positive definite function, okay. This cannot be 0 anywhere, but at the origin, okay. So if C is just negative, just a negative constant, then v dot is negative definite, okay. And then we have asymptotic stability, okay.

So remember this is not exponential stability, yeah, because our class K functions here were of this kind and the class K function here or the class Kr function here is of this form, yeah. So this is x1 4 plus x2 4. Basically, it is higher order. So basically what if you remember for exponential stability, we need three same order class K or class Kr functions. In this case, for the v, I have one kind of class K function, class Kr function and for the v dot, I have another kind of class Kr.

These are not the same order of magnitude because if you remember I gave you the example, if you increase the power, then I cannot have a comparison like this. You cannot make the comparison with constants, yeah. This will never work, alright. So not exponential stable, not exponentially stable, but uniformly globally asymptotically stable or globally uniformly asymptotically stable, however you want to say, yeah. Many folks use UGS, many use GUAS, so whatever, whatever is okay.

Great, so this is relatively simple example. What happens if c is positive? Then it should be obvious that v dot is actually positive definite, okay. Actually it does not mean anything until now to us. We have not done any instability results as such, yeah.

We just defined instability. We just mentioned that if the system is not stable or not uniformly stable, it is unstable, okay, but we have not given any tests or theorems corresponding to instability, okay. That is actually what are the assignment, yeah. So instability theorems that will be actually part of the homework, okay, so that you have to find an instability theorem, okay. So there do exist instability theorems and it does turn out that the system is unstable. I am giving you that, yeah, but you have to prove it using certain results, okay.

These results look very much like Lyapunov theorems, but they are not coming from Lyapunov himself, they are from some authors, some subsequent authors, okay. Alright, great. Any questions in all this material, Lyapunov theorems and stability and so on? Yes.

Yes. Yep, absolutely. That is always the case. All the results are purely sufficient results. So yes, yeah. I do not think it is exponentially stable, but yeah, but I cannot say conclusively, I agree.

Alright, true. Alright, great. So today what I want to do is what we promised that after we have looked at the Lyapunov theorems, after we have looked at some examples, we wanted to, you know, talk about the proof of some of these, okay. So we want to look at the proof of some of the stability theorems. So again I will say that the proofs are sort of motivated from Vidya Sagar's book, to some extent Khalil's book, to some extent some notes I found online, mostly because different sources because I have tried to distill it into a simple enough looking proof that we can follow, alright. That is the only purpose. Otherwise, yeah, you can pick up any good text in nonlinear control and you can expect to find this proof, alright.

I prove only two results, a stability result and an asymptotic stability result. Everything else is assignment, alright. They are very simple because once you have done these, it is doing a little bit beyond it is pretty easy. Yeah, I can promise you that.

Alright, great. So proof of stability theorems. The first one that we try to prove is the stability in the sense of Lyapunov result, okay. What does it say? So now I give you the complete statement because earlier it was split into pieces. So here we try to make the complete statement, okay. So what does it say? The theorem itself, there exists a V which maps time and states in some ball of radius r to real numbers for some positive r such that I have V t0 is 0 for all time.

I can even say that, right. And then I have V to be a C1 function. Yeah, so continuously differentiable function. C1 is continuously, once continuously differentiable. So the derivative also is expected to be continuous, okay. So once continuously differentiable and positive definite, then if V dot is negative semi-definite, that is V dot is just less than equal to 0 just as a function for all x in Br again.

Yeah, remember this entire analysis and all the trajectories we are expecting them to be in Br. So all of this, so even the V dot less than equal to 0 is in the same domain. If I escape the domain, then there is an issue. Yeah, of course V dot could be negative beyond the domain, but I don't care because I am not going to use it. So the point is within the domain it has to satisfy because otherwise I have a problem.

I might escape the domain and then I don't even have positive definiteness of V, then I am in some soup. So V dot is also negative semi-definite in the domain. Then we say that 0 is a stable equilibrium in the sense of Laplier, okay. So this was the theorem, right, just written in complete detail, nothing different from what we have already seen, right. I hope all of you are very clear on this statement already because you have seen it a couple of times now.

So how do we prove it? Remember the stability definition. I have already marked it here. So we have to prove it via stability definition. There is nothing else we have.

We just have to prove this definition holds. If I have this theorem to be, if I have these conditions to be satisfied, I want this stability definition to hold, okay, for this general non-linear system. Yeah, I have not assumed anything here. So x dot is f t x t. So yeah, I have assumed a very very general non-linear system, just that I have these sort of conditions holding to, okay. So very, I mean, so pretty powerful actually if you look at the result itself because I never have to solve the system.

And I still want that this condition holds. What is it? The stability definition says again for your benefit that for all epsilon positive there exists a delta which can depend on epsilon and t0. I am only proving stability. So it is allowed to depend on initial time, okay. Initial time is not a problem. Such that norm x0 is less than delta implies that the solutions lie within an epsilon ball for all time greater than equal to t0, okay.

We will start with the simple case and move on. What does it mean for us, for V to be positive definite? It means that the initial condition is 0, V is C1. This is already part of the assumption, yeah. Then positive definiteness requires that V dominates a class K function of norm of x, right. Alright, this is what is the meaning of positive definiteness.

And we want of course this is a class K function. So we require that it is also 0 at 0, it is also continuous and it is strictly increasing. Alright, okay, great, great, okay. So this is just to refresh, all these are refreshers, okay. We already know all this. I am just writing it out so that when we do the proof we do not have to you know go back to the definitions, okay, excellent.

So I am going to now do a stage wise proof. First do a very simple proof. I will say that let's assume this phi norm of x is actually alpha norm x square, okay. I will just assume that this happens. I am just saying that the class K function has a very nice structure, yeah, that is actually equal to alpha norm x square for some alpha positive, okay. So I am just assuming a very simple structure here, okay, just to make our life easy again.

It could be any class K function but I am saying it is actually alpha norm x square. So I am saying that because this is the case, then how can I do the proof because this will give me a nice hint on how to do the proof for the general case also, alright, excellent. So I am saying that V t x is greater than equal to alpha norm x square for again x lying in this ball B R and for all time this is the positive definiteness condition, right. This is just the positive definiteness restated for the simpler class K function, okay. What about this V dot? It is just negative semi-definite, right.

It is just saying that V dot is less than equal to 0 for all x in B R for all t greater than equal to 0, okay, nothing too complicated, just the negatives. So I have stated everything I have from the Lyapunov theorem, alright. Of course there is continuity and all but other facts are stated here, right. Now what do I do? I start with an epsilon, the user is giving me an epsilon, from that I construct an epsilon 1 which is the smaller of epsilon and R, okay.

Why? This is obvious, okay. Why do I do this? Why do you think I do this? Because we want to have a radius epsilon to remain in the ball of radius R. Absolutely. I have to remain in the ball of radius R, okay. Therefore, if you give me an epsilon ball which is larger than R, it does not make sense for me to consider that large epsilon ball. It is better for me to consider R itself, yeah, because it is irrelevant for me to use the large epsilon ball because R is smaller, right.

So I will just take the R ball and work with the R ball because my trajectories have to stay within the R ball, okay. That I am ensuring and guaranteeing anyway, okay. So therefore, I don't work with epsilon, I work with epsilon 1 because it is obvious that if the states are within epsilon 1, then they are within epsilon anyway.

So I am done. I don't have to do anything more. It is actually a better result if you think. Okay, great. So what do I have? For all x in B R, alpha norm x square is less than equal to V t x. This is just this written in the flipped form and this is less than equal to V t 0 x 0.

Why? Why do I get this? V dot is less than equal to 0. So V as a function of time cannot be increasing. So if I wrote this a little bit more carefully, this highlighted guy, it will actually be V t x t is less than equal to V t 0 x 0. So the left hand side is a function of time, right? And I have mentioned that this V dot is less than equal to 0. So along the trajectories, V dot cannot be increasing, okay. So whatever is the value of V at initial time, its value at any time beyond initial time, has to be less than equal to this, cannot be increasing, okay.

So this is simply coming from this. So this one sentence now codifies both our results, okay. And this one sentence is enough for us. Now what do I do? I forget the middle thing.

This is useless for me. So I just look at these two because this is in terms of initial condition, this is in terms of final state.

So I have alpha norm x square is less than V t 0 x 0, right? And from here I can get a bound on norm x square. What is stability? It requires me to bound norm x. So I have already come to a stage where I have a bound on norm x square. Yeah, pretty cool.

Now what do I know? What do I want? I will already state what I want. Forget proving it in the linear. So I don't like to do this linear proof because nobody can follow anything. That is why I did those steps. Now it is easy to follow. If this is less than epsilon 1 squared, epsilon 1 squared, if this guy is less than epsilon 1 squared, I am done, right? Because then I have norm x square less than equal to epsilon 1 square.

So norm x is less than epsilon 1 and I am done. So what do I want? I want this, whatever is in the green bracket. So my work is cut out because if this happens, norm x square is less than epsilon 1 square which is less than epsilon square and I am done, stability done, right? Now you will remember that I have not chosen a delta yet and that is what is going to come out of this guy. That is what is going to come out of this guy because you see the left hand side is depending on initial condition and also time t0 and the right hand side contains an epsilon. So somehow I have to be able to solve for x0 from here, okay? That is the whole idea. So once I do that, I will be able to get an epsilon, okay? And my claim is it is, so because from here I need v t0 x0 to be less than alpha epsilon square, just rewriting this guy.

My claim is that I can choose delta, I will choose delta such that this happens. That supremum over x less than delta, norm x less than delta v t0 x is less than equal to alpha epsilon square. I am saying this is possible. I can choose such a delta, okay? It is not giving you how to choose but it is saying I can choose such a delta that this happens, okay? Thank you.