Nonlinear Control Design

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Week 3 : Lecture 18 : Lyapunov stability Theorems- Part 6

We have seen a few examples already, so this is one very relevant example, can anybody recognize this system, do you know what this system is? What is this dynamics of? Yeah, this is a damp pendulum, standard pendulum that we will see invariably they are damped, there is friction at the joint, yeah. So I cheated because otherwise I would have to do hard work like this to prove good things, okay, I would have to construct functions like these and all, yeah, that is why I have introduced another term here, right. If you see typically a pendulum will have dynamics x1 dot is x2, x2 dot is minus sine x1 minus x2, so it is a non-linear oscillator actually, it is a non-linear oscillator. So, but I have introduced this minus sine x1, yeah, because it will make my analysis simpler, yeah, otherwise basically it becomes complicated to follow, I mean I can do the analysis but you will just say why are we talking about such complicated things in class which you cannot follow, that is more of an assignment thing that you can try, alright. So, what is the Lyapunov candidate in this case is this guy, anybody recognize what this is? Anybody, what is this? For the pendulum system what do you think this is? This function 1 minus cosine x1 plus half x2 square, hmm energy, exactly, just think pendulum, x1 is this theta angle, so 1 minus cos x1 is the potential energy and half x2 square is the kinetic energy, x1 is theta, x2 is theta dot, so theta dot square half gives you the kinetic energy and 1 minus cosine x1 because cosine theta gives you the potential energy, exactly the energy of the system. So, don't look at me funny, how did you come up with this thing, how will you do this in the test, this is the energy of the system in this case, okay, alright.

The more complicated thing is to discuss the definiteness of this, hmm, I hope it's evident to you that this becomes negative, does this become negative? No, doesn't become negative, so one thing is evident it is 0 at 0, yeah, x1 0, x2 0 this is 0, good thing, because 0, 0 equilibrium is what we are thinking about, yeah, this is the equilibrium, the downward equilibrium is what we are interested in, great. Now, what happens if I take x1 to be arbitrary, is that notice how do we check for positive definiteness, we check where it becomes 0 and if any of those points are non-zero states then we have a problem, okay, so this becomes 0 whenever 1 minus cosine x1 becomes 0 and 1 minus cosine x1 will become 0 at all 2n pi, all 2n pi 1 minus cosine x1 will become 0, so for the simpler case at x1 equal to 2 pi which is this position, this is theta equal to 0, this is theta equal to, no, not this position, coming back to this position, yeah, so see this is another thing about using Lyapunov functions and working in Euclidean space, actually the system is not in really Euclidean space, yeah, because this is an angle, yeah, it is just when you say 2 pi physically it is the same configuration, but the Lyapunov function and the system doesn't recognize all this, okay, does not recognize that this is not the same configuration, so when I look at x1 equal to 2 pi, this is still 0, so for all states of the form 2n pi, 0, this is 0, so this is not positive definite globally, yeah, like Vidya Sagar likes to say, LPDF and PDF, this is not PDF, this is only LPDF, that is locally positive definite, actually we have only defined locally positive definite, now this is the case where we have to define a Br, a ball of radius R around the equilibrium, what is the ball of radius R? We want x1 to lie between this and x2 can be anything, okay, this is a funny looking ball, not a ball at all, but I hope you understand, there is something local about this, yeah, in one axis if you draw it, it is like in one axis it is only this much, in x2 axis it can go anywhere, this cylinder actually, cannot go further here, so minus pi and pi here, but in x2 you can do whatever, okay, fine, yeah, this is x1, okay, if you are not comfortable with this cylinder, you can take x2 to inside whatever minus R to R and you are fine, you can actually make a ball, yeah, this makes you uncomfortable, this is fine too, okay, so inside this region, minus pi cross pi, minus pi, pi cross R, notice the end points are not allowed, yeah, the important thing to remember whenever I make this is that the equilibrium you are interested in should be within this, so I mean none of you asked me, but this was also valid choice, right, this is a valid choice for when this is positive definite, 0 is not included, okay, so this is a sort of a problem, yeah, and of course, so this is not okay, so therefore, minus pi to pi is the only reasonable choice, you can think about it, yeah, and remember whenever we make neighbourhoods or balls, they have to be open sets, we discussed this, I hope you keep this in mind, therefore, this has to be open, can't just say 0, 2 pi, s and a, not like that, so it can't be 0, 2 pi open close or anything like that, it has to be open at both ends, has to be an open set basically, so this you will see that this is the only reasonable choice because inside this set, 0 is the only point where V is going to be 0 and everywhere else, it is going to be strictly positive, okay, alright, great. So, once you verified this fact, this is the only problematic thing, you can see it's already decrescent, it's free because there is no time argument in V, so V is already decrescent for free, you don't have to do any special work there. Now, if I compute V dot as always, I will get sine x1 x1 dot from here and x2 x2 dot from here, okay.

Now, I substitute for x1 dot, so sine x1 and x1 dot, I substitute here and x2 x2 dot, I substitute here, alright, again not doing anything too complicated, I am simply substituting the derivatives, in this case because I have made a hack, everything turns out to be nice, we will see, we will see. Now, if you see this term and this term cancels out, yes, alright, so I am left with minus sine square x1 minus x2 square, yes, minus sine square x1 minus x2 square, both are square terms, so already I am feeling good, yes, alright and now if I look at, now I have to test the negative definiteness by the way, my domain is fixed, I can't change this now, just for V dot I can't come up with a different one, I am restricted to the same Br, whatever that R is, so in this case I have chosen this open set, yes, so I have to stick with this, I know that this is always negative definite term, no problem, where can this guy be 0? At x1 equal to n pi, x1 equal to n pi, ok, so the only possible candidate within this set is 0 itself, yes, because minus pi and pi are not in the set, alright, so this guy is exactly 0 only when x1 and x2 are exactly 0, not 0 anywhere else, so I really hope you are able to capture these certain points here, very certain but very key, if you miss this your analysis is wrong, if you don't give me this minus pi, pi set, you cannot prove anything at all, that's completely wrong, and if somehow these terms come out so that in this minus pi, pi set there are multiple places where this is 0, that is also a problem, if something like let's see, divided by 2 kind of a thing happened, then what would happen? Suppose I ask you, just to test, now what? The analysis will go through, will it go through? Good point, will it even go through? This will become sin x1 by 2 times half, there will be a half here, and x2 x2 dot will be half here, half here, half here, and a half here, this will also have a half, now what happens? Analysis will not go through because of the this term, but I can always cheat, I like doing that, so I will make this half, okay, I just played with the, I made things half so that analysis will go through, I have just changed the dynamics a little bit, there is minus sin x1 by 2 now, okay, so the analysis will go through, I can promise you with this dynamics now. Now what about positive definiteness and so on, where is it positive definiteness now? What is the Br? All I need is the ball Br, what will it be? minus 2 pi to 2 pi, how are you computing it? So you want 1 minus cosine x1 by 2 to be 0, you are looking at where that is 0, so you want cosine x1 by 2 to be 1, okay, where is that? x1 by 2 has to be what? 2 n pi, 2 n pi is that correct? Okay, so 4 n pi, 4 n pi, okay, so or if you think of it as, it will be, if you think of it as plus minus, then it is plus minus 2 pi, so basically this set will become minus 2 pi to 2 pi, okay, so this analysis is sort of expanding things, okay, expanding where you can work with minus 2 pi to 2 pi, which is more or less going to cover everything I guess, minus 2 pi to 2 pi will have 0 also and will also have pi, okay, funny, so this makes it nicer actually, if I do x1 by 2, so if it so happened that your harmonic oscillator had x1 by 2 instead of x1, then this pi thing, so the pi thing will not be in equilibrium at all, right, if you look at the equilibrium of this guy, what is the equilibrium? x2 equal to 0 is an equilibrium and sine x1 by 2 is an equilibrium, has to be sine x1 by 2 equal to 0 is an equilibrium, so either x1 has to be 0 or x1 has to be 2 pi, which is the same as 0, so if I modified this pendulum equation in this way, then this inverted position is not an equilibrium at all, it's only this position and this position, okay, is that clear, just by making x1 to be x1 by 2, okay, I'm going to erase this, sorry, alright, this is just to… minus pi to pi will still work, the only thing is you have, yeah, I was expecting a more shorter, more constricted region, turned out to be the other way round, but minus pi to pi will still work, the only thing is you have given a smaller range, right, I mean, see, whenever you talk about local, when you say that it's stable, asymptotically stable in this case, it turns out to be asymptotically stable, right, of course, it's not global, remember, non-linear system not global, in fact, in this case, pi is also an equilibrium, remember I told you, if you have multiple equilibria, then global is not possible, just like in optimization, so it's fine, but whenever you give local results it is, you want to give as large a region as possible, not the smallest one or not any one, you try to get the largest one, okay, okay, alright. Now there is another example here anyway, so this is, but before that I want you to remember that there is also notions of conversely which essentially says that there exists a Lyapunov function for every stable system, okay, so remember in this Lyapunov theorem I mentioned that this is a candidate Lyapunov function, once this candidate Lyapunov function satisfies any one of these results, then it is called a Lyapunov function, okay, so this is a candidate satisfying any of these, it becomes a Lyapunov function, okay, so what does the converse Lyapunov theorem typically says? It says that if the equilibrium is stable then there exists some Vt x positive definite C1 such that the Lyapunov theorem is satisfied, okay, so there do exist converse Lyapunov theorem, the problem is they are not constructive, it's not that using the theorem you will be able to construct a Lyapunov function, yeah, so it's as good as, I mean it's a nice mathematical result but it's not going to help you construct anything, okay, so actually so you can think of it as an if and only if condition, the problem is you will never be able to get a V out of this converse, so therefore you are still left with trying to hunt for the best Lyapunov candidate possible, okay, any questions? Yes, yes, what will leave the minus pi to pi bound? Yes, yes, absolutely, it's a very good point, so whenever I, remember whenever I give this ball of region R, ball of size R or whatever domain I give you, yeah, or the domain you come up with to ensure positive definiteness of this V and so on and so forth, yeah, you implicitly assume that your system trajectories always remain within this ball, okay, this is an implicit assumption, it's not being given to you for free or anything like that, so yes, those are not allowed, those are excluded, okay, so yes, if you want you can make this R to be something smaller, yeah, but then that will depend on, I mean you can understand, right, I can't solve this system to actually come up with that R, would be virtually impossible, I mean this is the simplest nonlinear system but you know if you get to even little bit more, you still, definitely won't be able to solve to get such an R, so it's an implicit assumption that this R is there, but then again in a lot of these cases just like here, if this was any BR, yeah, all these results would go through, because the second term has no impact on definiteness, so for any bound you choose on your theta dot, the results will go through, so it's sort of you know, you will still get your stability is what I'm saying, but yes, I do agree that you will get out of this, but one of the other things that I would like to also say is that I'm not missing anything here, right, by doing – pi and pi, pi is the only one that I'm missing, and there is no choice but to miss pi, remember without missing pi I'm including the other equilibrium in my analysis, there is no way you will be able to prove even stability after that, because now your domain contains both equilibrium, yeah, so missing pi is not a choice, I'm very carefully missing pi here, so that this equilibrium, because you know for sure that if I start at this equilibrium and there is no disturbance or anything, yeah, then it is never coming back here, it's staying here, therefore no question of stability, right, so therefore this is a problem point and so I have to have to have to miss that point, so yeah, if your velocity makes you go through it, then you have to hunt for what we call semi global type of results, yeah, which will essentially say that, okay, you sort of, you will not actually stay there because there is disturbance and you are very, even if you are arbitrarily away from it, you will fall back and so on and so, so we will say things like that, yeah, but yeah, that's also an important point, okay, alright, anything else? Alright folks, we'll stop here, thank you. Thank you.