Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 3 : Lecture 17 : Lyapunov stability Theorems- Part 5

So, let us look at another example, ok. So, I took the same Lyapunov function as you can see that was my aim, because I can keep the same Lyapunov function and come up with a different example that was the plan, alright. Great. What is this? And we already know this is continuous positive definite and radial unbounded and all the nice things, huh? Not decrescent. Now, V dot as usual is x1 x1 dot 1 plus t x2 x2 dot and x2 squared by 2. Here I have just used the chain rule, sorry product rule, product rule, not the chain rule, the product rule.

I always get confused between the product and this is not the chain rule, it is the product rule, ok. Chain rule is for chain of functions. And then as always I substitute the dynamics. This remains the same, yeah.

Here 1 plus t x2 is as it is, then I have substituted for x2 dot right here, huh? Because of this wicked choice of system that I have made, the first term will cancel this term, right. If you see 1 plus t 1 plus t cancels out and then the first term will cancel this term, alright. That was the plan, huh? Yeah. And then the second term again by my wicked choice of system will cancel this term, alright. That is exactly how I chose it.

This guy multiplied by this is minus x2 squared by 2 and this is plus x2 squared by 2. This was what was remaining last time. So I made sure I cancelled it. Yeah, ok. Is that clear? Notice this is all that was remaining here.

So I just introduced the term here to cancel this, ok. Cancelled. Now I have v dot to be exactly 0, ok. v dot to be exactly 0, yeah. Very ridiculous looking system, huh? But it is stable at least, alright.

It is stable. It is not, v is not decrescent, so cannot claim uniform stability. Again I am saying not uniformly stable but it is more appropriate to say cannot claim uniform stability with this v, ok. If you do not get a property from a v, does not mean that the system does not have the property. That you have to conclude in different ways.

I am most certain that this is not uniformly stable, huh? But it is not obvious just by choosing one Lyapunov function and doing one v dot and claiming. No, that is not enough, yeah. In fact if you do 100 even then you cannot claim, yeah. It is just a deficiency in your choice of v and not in, you know, in terms of the system itself, yeah, ok.

Great. Any questions? Alright. So it would be very illustrative to see the difference between these two systems, ok. If you look at this guy and if you look at this guy, ok, forget how I modified it and whatever did, whatever just to cancel terms. This was just a modified harmonic oscillator, right, because if this term was not there or for small time just like we discussed, it is a pure harmonic oscillator, ok. And this system is actually a damped harmonic oscillator.

It is a modified damped harmonic oscillator, ok. Why damped? Because of this term. This term is acting as a damping, right. If you, all of you folks who do PID control know that the derivative term is the, they are all connected. So derivative term is the damping term, yeah.

So this is the derivative term. This is the proportional term, this is the derivative term. There is no integral term here but honestly speaking in typical non-linear control design you don't get a integral term. It would be rather unusual, yeah. But there is a PD control if you think about it, yeah.

But with a time varying modification here, yeah. So this is a modified damped harmonic oscillator, ok. If time was small or if this term did not exist or it was one, then this system is actually what? You mean, I mean in Lyapunov terms, what is this system in terms of Lyapunov stability terms? Uniform what? What do you know about this system? About a damped harmonic oscillator. What is the example of a damped harmonic oscillator? Spring mass damper, alright.

Ok. So if I leave a spring mass damper from any arbitrary initial condition what happens? Come to rest at origin unless your spring poor quality and they start stretching and all, yeah, which is the case in general. But yeah, they come to rest at the origin. So what does it mean? What does it mean about stability property of the system? Asymptotic stability. It is asymptotic stability, yeah, in fact uniform because of no time dependence, yeah. So if this was not there, this system is in fact uniform asymptotic stable, alright.

So very strong property. In fact it is exponentially stable. Why? Why do I claim it is exponentially stable? Yes, yeah, I am looking at you. Why am I claiming it is exponentially stable? Without doing any analysis. All I talked about was asymptotic stability by giving an example of leaving a mass in a spring mass damper.

Why is it exponentially stable? Yes, we discussed this. For a linear system asymptotic stability, exponential stability is the same. We proved it.

We proved it. Ok. Well, we did not prove it. Did we prove it exactly? Yeah, we proved it, right? Yeah, we proved the stability one but from that you can see that the exponential stability one is also proved. Basically any convergence for a linear system is always exponential. A linear system will never converge at any other rate, ok, than exponential, linear time invariant system.

Ok. So, but as soon as I introduce this time dependence, I drastically reduce what I got. I am now only stable, not even uniformly stable. I am quite certain it is not uniformly stable. Ok. So significant drop in what I can do now.

It is very simple to see why things go wrong. As time becomes large, this stabilizing term, this so-called PD controller, if you put a PD controller on a system and after a certain time you stop the controller, you make the control zero, ok, then the system is not going to be stable anymore, right? Ok. Here it works in the absence of the time dependence it works because it is perpetually acting. As soon as you introduce this time, the effect of the proportional derivative term is dropping at a linear rate, ok. So which means that your proportional derivative term which is what is stabilizing the system is dying, ok.

So the proportional term gives you stability, the derivative term gives you asymptotic stability, yeah. So these terms are, the contribution of these terms are dying as time increases and therefore the system will not be stable anymore. So you lose all these properties, ok. So at best what you get is stability. In fact you are lucky that you at least get stability, ok.

Alright, any questions? Yeah, if you think about the control gains, this is the gain, right, 1 over 1 plus T, right, Kp, Kd, whatever if you think about it. The gains are going down, right? So if you ever design a PD controller, PID control with your gains going down, you will see the system will not work, right? It's obvious, ok. Alright, great. So let's go to this R guide, yeah, when the time dependence is removed. I already said that this is exponentially stable, ok, because for linear systems asymptotic and exponential is the same.

In fact in this case you can even solve this system, right? This is not difficult to get a solution for. The solution will have exponential decay term, so it's exponentially stable. But if I wanted to do the very very hard work, it turns out to be very hard work of proving exponential stability via Lyapunov functions, then it's very complicated. That's what I have sort of illustrated here. But then sometimes you do need to do this kind of an exercise because your system may not be linear always.

So I cannot choose something as simple as x1 squared plus x2 squared by 2 as my Lyapunov function. That does not work. I can promise you it does not work. It will not work. So I have to choose something like this, ok.

Later on we will go into the motivation of choosing something like this and why something like this might make sense and so on when we do design. Yeah, so I am not telling you anything about why and how I choose it, ok. I am just choosing it, yeah. So this is a little bit complicated. But the idea is basically based on the notion of backstepping, ok.

But we have not done backstepping yet. So don't worry about it. This is just, think about this as just an example, alright. Alright, so what is this? Half k x1 plus x2 squared plus 1 over 2 alpha x1 squared. So it's not just a combination of two terms. I have basically introduced some random constants also, where this k and alpha are positive constants, ok.

I know that this is positive definiteness and definite in fact radially unbounded. Do you understand why? Why do you think this is positive definite and radially unbounded? See all I need to verify for positive definiteness is what? That for non-zero values of the state, it is strictly positive. So what about, do you think this is? k and alpha are also positive. So is it that easy? Ok, but if you remember we had discussed terms, V terms like x1 plus x2 whole square and I said it is not positive definite, right. Because this is also zero when k x1 equal to x2 is minus k x1, right.

So there is a problem, right. I mean because you guys immediately said yeah yeah it is positive, everything is positive. Then what does it do? These are not good arguments. No no no no no.

Can't be so vague. No no no no. Don't do that. To be more precise, you tell me a little bit. This is where I am, I may not ask you to design these things but you have to be very precise about why this is positive definite.

We have already discussed this. Wait. I am going to write this again. This. Now what are you saying? Both terms are positive.

Both terms are positive. You mean non-negative. Yeah. Then? Ok. So when are they zero? Which means that individual terms are zero when x2 is minus k x1 and this is x1 equal to zero. And this is an and condition which implies x2 is also zero. So the only place where this, so things change because I added this x1 square.

If I did not add this, did not have this, then this is not positive definite anymore. Because it is zero on a line. And that's not positive definite. Ok. So only because I added this, so there are now two conditions need to be satisfied.

This has to be zero and this has to be zero because they can't cancel each other. These are all non-negative terms. Therefore this and this both have to hold which means this happens.

Ok. So be very precise. When I ask you, don't just say positive, positive, positive, ok. No. Otherwise why did we do all these definitions? There was a very good reason.

Ok. Of doing these definitions carefully. So please be careful when you state that something is positive definite or not. Alright fine. Sounds good. That's what I have said.

So anyway the k is missing here, it doesn't matter. I will put a k here.

Ok. Alright. Great. We take the derivatives. Yeah, just like we are doing. Pretty straight forward.

Right. Lot of bookkeeping. In this case it looks complicated. Yeah, but eventually I am not doing anything very fancy. I am just taking derivatives and substituting for the derivatives. So here I will get kx1 plus x2 times kx1 plus x2 derivative. I will get x1 and x1 derivative.

And the half will go away, this half will go away. Ok. That's it. Alright. Now I substitute for the dynamics here. Kx1 dot is kx2, x2 dot is minus x1 minus x2.

Similarly this is x1 x2. Alright. Now we have to do a lot of manipulation. Yeah. Kx1 plus x2, kx2 minus x1 minus x2 and then I write this as x1 times kx1 plus x2 minus kx1 square by alpha.

Ok. So this is just equal to this. Do you believe me? Yes. I have just added the kx1 term. And why did I do this? Again these are manipulations that you will have to do. These are the kind of manipulations you will have to do in any Lyapunov analysis. So better be comfortable if you are not asking why did I do this? Why do you think I did this? See in the harmonic oscillator case this guy was getting cancelled directly.

Ok. Now I know I cannot cancel it directly. I have a problem. So what do I do? I do the next best thing. I try to club it with the term I have already. And I see that here I have a kx1 plus x2 term. So I try to write it as kx1 plus x2.

So what did I do? I took the x2 I wrote it as kx1 plus x2 minus kx1. So I got from this one term these two terms. Now I know that this guy can be clubbed with this term.

Ok. Alright. I can't cancel so I try to club them together. This is the only few manipulations you can do. Ok. There is one or two more which we will get to soon. But right now remember this is the only thing I can cancel or I can club terms.

So I cannot cancel. So I try to club it with the term I already have. So this is the only term I have. It doesn't make sense to club it with this. This is way more complicated anyway. Why would I do that? I will just club it with I will just introduce this term here.

Ok. And the cool thing that happens when I do this. This is again a product of the backstepping method which we have not discussed. By introducing the kx1 and a minus kx1 I actually got a negative term in x1. Yeah.

Both k and alpha are positive. So this term is actually a nice negative term. It's a good term. It's a helpful term.

Ok. Now this guy gets combined with this. And what do I have? kx1 plus x2. kx2 minus x1 minus x2 and x1 by alpha.

Right. This is coming from here. Alright. Make sense? Ok. This first term is coming just by the clubbing of the terms. Alright.

And I already have a good term here which I keep as is. Yeah. I love the good terms. As soon as I get negative quadratic terms I keep them as it is. Never touch them. They are what will help me eventually.

So I never touch them. This gets carried on until the end of the analysis. Now I have only two variables actually x1 and x2.

Yeah. So I just club all the terms in terms of x1 and x2. Ok. Alright. So I get something like minus 1 over 1 minus 1 by alpha minus 1 minus k and all this mess.

But I get some terms. Ok. So now what do I do? I take this 1 minus k common. I pull it out. Yeah. These are all variables that I choose.

That's why I kept these handles or knobs. Ok. So that I can play with them. This is for me to gives me freedom to play with these. Yeah. Otherwise I will not be able to choose a good V.

Yeah. When I was starting to choose a V I didn't have any idea what is going to be k and alpha. But once you conclude the analysis you will see that the choices of k and alpha will become obvious. Alright. So I have deliberately written everything as negative terms.

Why? Because V. needs to be negative definite. So writing it as positive terms is ridiculous. Makes no sense. So I have written everything as negative terms. How will I get V. to be negative definite? I want this term and this term to look identical.

This should also look like k x1 plus x2. Yeah. I am doing this carefully. Just follow the steps. Ok. These are things we do often. So I take 1 minus, minus 1 minus k common outside. Now the first term is the same. The second term has 1 minus 1 over alpha divided by 1 minus k plus x2.

And this is of course my favorite term. Remains as it is. Now what will I say? I will say that I want my k to be equal to this guy.

Yeah. And if it is, suppose it is. Yeah. Just to it, it is equal to k. Then these two terms together become k x1 plus x2 whole squared.

Right? And this is some nice negative term. This is nice negative term. I have negative definite V. Ok. Now all I need is this has to be equal to k. Ok. And of course I want k to be less than 1.

That's the other requirement. Otherwise k has to be strictly less than 1. All good. It is my choice. It is just a V. Right? It is just some function I use for analysis.

It is not changing anything. Alright. Now you just have to see if this is feasible or not. Just have to check the feasibility of this guy. So k is between 0 and 1.

Alpha is positive. I want to check the feasibility of this. This will give me a quadratic equation in k. Ok. Which is going to solve this.

I have to choose one of them. Yeah. I can choose any one of them. Right? They will both satisfy. So if I choose k as this guy, I am fine.

Yeah. I can choose actually any one of them apparently. Alright. So I think I took an example or what. Let's see. Yeah. So first I am trying to ensure that this, because that is what is going to give me the feasibility. I want to check that the discriminant is going to be positive or not so that I get a real outcome here.

So that's all I need to do. I get alpha less than 4 by 3. Ok. Because this is a linear requirement. Yeah. So alpha is positive but less than 4 by 3. Anything less than 4 by 3 is good. Ok. Now if I assume that this is equal to half, some value less than 4 by 3, then I get whatever.

Then I get alpha is 8 by 7 which is fine. No problem. Yeah. And I can choose k as any one of them. Actually it has to be less than 1. So I will choose the, it's preferable to choose the, let's see, negative one.

I am sorry I can't erase this can I. I will just choose the negative sign. Right. Yeah. I will just choose the negative sign because whatever appropriate value of alpha I choose less than 4 by 3, you see that the quantity inside is going to be less than 1. Because it is 1 minus some quantity less than 1. It's something less than 1 so square root of that is also less than 1. So it's better to choose the 1 minus because if I choose the 1 plus, 1 plus will also work for some time because I have a divided by 2 and all that.

But this will be an easier choice. This is guaranteed to work. Yeah. So basically I have given you a choice of a k and a choice of an alpha. Yeah. So alpha is exactly this and so on and so forth.

I mean whatever. Alpha is exactly this if you want something like this and k comes from here with this choice of alpha. Alright. So with this very very complicated construction I have proven exponential stability of this system which is very easy. Yeah. See in this case it does take a lot of work because the first, because exponential stability requires the same order of magnitude functions and so on and so forth.

Alright. So if you look at this guy and you look at the V dot, these are exactly same looking functions. They are the same order of magnitude functions. They both have the same quadratic terms kx1 plus x2 square and x1 square.

Yeah. So I actually got same order of magnitude functions. Yeah. And so by my Lyapunov theorem it is exponentially stable. Yeah. So this is not too easy to do for non-linear systems in general. Okay. Thank you.