Nonlinear Control Design

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Week 3 : Lecture 13 : Lyapunov stability Theorems- Part 1

If you remember in the last week's lectures, we were looking at stability. We ended with interesting examples alright of stability that is attractive but not stable, asymptotic, asymptotically stable, globally asymptotically stable, non-uniform stability and so on and so forth ok. And then finally what we had done was we had sort of proved some simpler results for linear systems ok. So what were the simpler results? It sort of characterized stability in the form of bounds on the state transition matrix ok. But in spite of all of this we sort of explained and I hope all of you has understood that this sort of characterization is still a definition of stability ok. It should be considered a definition of stability and not necessarily a test of stability because using these definitions is not, it is very easy ok.

For time varying linear systems you will still require to solve the system in order to get state transition matrices in order for us to be able to talk about boundedness ok. So these are still to be seen as definitions and not as tests and that is where we are going to move on to this week. So we start with our material on Lyapunov theorems ok. You can see that I am using my earlier NPTEL node.

So these parts are there is some commonality here. Of course there is a I mean we will look at a little bit more detail on some proofs but that will come probably later this week or actually it won't come this week it probably come next week ok. So like I said I am out on Friday. So we will not have a class on that. Alright so let us first look at the results alright.

We just look at the results how to use them and we will look at the proof of you know one or two of these results later ok. That is the idea. That way we are you know we have some handle on what we are even trying to do ok. So as I had mentioned very clearly I think that there is a problem with stability definitions we have to solve the system of equations ok. Now this is has to be the case because if you are using epsilon delta of any form you will have to solve yeah we whatever examples we took we actually solved the system ok and which is not possible for most non-linear systems ok.

And so we are looking at actually not quantitative but I would say more qualitative methods yeah but anyway I will leave this word as it is because depends on how you think about it ok alright. So before we go on to talk about the Lyapunov theorems we need some background ok. So we first talk about function classes. All of you already know a few function classes by the way these are characterized by the LP norms the capital LP norms. So you already are aware of some certain function classes yeah these are additional function classes that we are going to talk about.

So I hope you keep in mind that there are several kinds of categories of functions that we tend to invoke in all our analysis ok. So what are these function classes? The first one is a class K function ok and all of these functions are defined from R plus to R plus ok. So non-negative reals to non-negative reals ok. R plus denotes non-negative reals so 0 is included in this set ok. So a function phi is of class K if it is continuous strictly increasing and it is 0 at 0 ok that's it only these three conditions ok.

Examples are like I mentioned here 1 minus e minus x and x. Why you can verify when x is 0 this is 0 right. So here also I have mentioned if you take phi x equal to x this is one of the simplest examples right because it is 0 at 0 it is definitely strictly increasing right you can take a partial and see and it is definitely continuous in fact it's smooth yeah. Similarly you can take any other polynomial you can imagine that any polynomial would work yeah polynomials are in fact analytic functions so way more than smooth yeah they are even beyond smooth. So polynomials all work yeah as long as the coefficients are positive and all the nice things are right otherwise it might be decreasing yeah that is a problem.

So typically if you take you know x square is also fine ok it's also an increasing function yeah because we are looking only at 0 and beyond yeah arguments are non-negative reals ok. So therefore, this works yeah. So this also works on the negative side, but if I take x cube then all then there is a problem alright ok. Then finally, we have a function which is of this kind 1 minus e minus x I keep using this example even for supremum I use this example actually turns out to be very nice interesting function. One minus e minus x if I take partial with respect to x it is what e minus x which is positive right whenever you know x is non-negative and as long as x is not infinity and infinity is anyway not part of real so we are fine ok.

So therefore, this is a also continuous function if it is smooth again also strictly increasing function and it is 0 at 0. So all three functions satisfy these criteria ok is that clear. This is what is the class K function remember arguments are always scalar valued non-negative reals and the output is also or the image is also scalar valued non-negative reals ok. So scalar valued functions are all we are talking about do not ever design a vector valued function or a vector argument function and think of it as class K no ok there is no such character. So, what is the function called class L function? Lot of a flipped version a function phi again you see that this never changes is class L if it is continuous strictly decreasing and it is the initial value is finite ok.

An example is 1 over x plus 1 because at 0 this is of course 1 it is strictly decreasing is evident because as x keeps increasing again remember x is in R plus right as x keeps increasing this keeps decreasing obvious and it is continuous continuity you can also verify continuity yeah because it is there is only continuity issue at x equal to minus 1 but that is not a valid argument so it is fine ok. One of the things that we need to note is that if a function is class K then it does not mean that the negative is class L ok. I hope this is sort of obvious because as soon as I take a negative my image lies in R minus yeah which in itself is not allowed ok. So, as a definition it is not allowed yeah. So, negative of class K function is not a class L function alright.

So, negative valued function we do not use yeah this is just a why we talk about these functions is these are what make up Lyapunov functions yeah and Lyapunov functions if anybody has ever seen these are actually like energy like functions and we never talk about negative energy not in our community I mean may be in quantum we can talk about it but I am not an expert there so I cannot comment on but yeah here we do not obviously have notions of negative energy. So, Lyapunov functions typically have connotations of energy so obviously we do not allow negative valued quantities here ok. So, class K and class L two characterizations now we have another characterization which is a sort of a stronger characterization it is a class K R function ok. A function again same arguments is class K R if it is class K and it goes to infinity as the argument goes to infinity ok that is it this is the only additional requirement ok. So, one of the examples we considered for class K deliberately was this and that function looks like this strictly increasing by the way no problem but it maxes out at one alright never exceeds one for whatever value of the argument right because plot looks like this it is getting closer and closer to one but never actually hits one yeah.

So, this is not class K R because even though the argument becomes infinite or tends to infinity we are all talking about tending to infinity your function value will tend to one in fact yeah so not infinity. So, these are there is a certain limitation about these functions that is why they are only class K yeah and not class K R but if you take this polynomial functions that we looked at they have all the nice properties right it goes to infinity as the argument goes to infinity. So, these are in fact class K R function ok. So, as we move on you will see that class K functions are connected with motions of local stability or stability class K R functions are connected to motions of global stability ok and class L functions are connected to uniformity ok. So, these are the three classes of function then the three kind of stability definitions we have seen everything else is a sort of combination of all this right because you say that you have some kind of you know stability uniform stability global stability global uniform stability and then there is the you know qualifiers of asymptotic and so on and so forth that is ok that also we will see how they are connected ok.

But local stability typically class K tested via class K functions global stability via class K R functions and uniformity using class L functions alright ok. So, one of the key things that anyway I have already mentioned is that we use we assume an equilibrium of 0 ok we assume the equilibrium to be 0 alright say that again. No R is just instead of using X I have just used R just the argument ok I have just changed the labeling on the argument alright. So, it is a class R function sorry it is a class K R function not that R ok this is just some notation for the argument yeah I have just replaced X by this yeah alright ok great. Now we are in a position to talk about definiteness ok once we have defined this class K function class K R function we can talk about definiteness ok.

Before I even go to definiteness I hope all of you we just looked at it you know couple of classes ago for matrices symmetric matrices you have very clean and clear notions of definition definiteness here you have this notation by the way whenever I use this notation this means that the matrix is positive definite right because there is no idea of positivity of matrix otherwise right. So, whenever I use this notation A greater than 0 it means I am saying that the matrix is positive definite ok. You also know that it means that the quadratic forms are always positive for non-zero X you also know that the eigenvalues of A are always positive and all principal minors have positive determinant yeah we exactly looked at these 3 characterizations ok for positive definiteness of matrices. Now what we want to do is generalize positive definiteness of matrices to functions ok because again those of you who had exposure to doing internal stability for linear systems you know that you use something like the Lyapunov equation right and the Lyapunov equation essentially looks like what it says that right where Q is and Q and P are symmetric matrices so given so the statement of course goes more formally like given a Q there exists a P which solves this Lyapunov equation yeah why this characterizes stability is because you choose your V a Lyapunov candidate in fact a Lyapunov function in fact as X transpose P X for the system right and if you take a derivative of this V dot and your system is governed by X dot equal to A X because this is the system for which you are studying the stability then V dot is actually turns out to be X transpose P A plus A transpose P X ok alright ok make sense and this yeah is in fact in fact I did not write it completely this is positive definite this is positive definite ok you actually require that given any positive definite symmetric Q you can obtain a positive definite symmetric P which satisfies this Lyapunov equation ok and once we have that essentially what we are seeing is that this becomes a positive definite function right it is a sort of extension of positive definiteness from matrices to functions this is now a function of X right. So, so this is what we want to connect now we want to say that we may have more general forms ok then X transpose P X something quadratic something simple non-linear systems have lot of different structure right you do not you cannot necessarily say that for every non-linear system I can use a quadratic Lyapunov function alright that is in itself a big assumption if you say that there exists a quadratic Lyapunov function for a non-linear system that is a very big assumption already because you are somehow saying that you can use linear notions to analyze this non-linear system ok so anyway so the point is we want to have a more general characterization of positive definiteness for functions ok great.

So, that is where we are going that is where we are going that is the aim here. So, definiteness what does it mean it says that positive definiteness requires that you have a scalar valued continuous function which is of time and of some states belonging to a ball of radius R that is this guy right this is what is the ball of radius R we have already spoken about it if you use different norms you will get different shapes here you can get in general you can think of it as an ellipse yeah or a circle but you can also get square and rhombus and what not depending on norm ok but it is a ball we keep calling it a ball of radius R ok yeah we otherwise you have to say neighborhood and all that yeah so that makes life a little bit more complicated alright Tk great. So, we want this function V and this is the most standard notation you will ever find of time and states belonging to a local region and it maps to real numbers ok it is always scalar valued so energy like again energy is the most if you think kinetic energy potential energy scalar value take in the state give you some scalar value ok that is the most obvious characterization of V yeah but remember in more often than not we don't use the energy of the system as V ok in several cases we do but many more times we don't alright ok what do you require from this? this is as of now I have only defined the you know domain and the range ok what do I require? I require that it is zero for zero states ok the function has to be zero valued when the states are at zero it is almost like a norm condition remember yeah has to be zero when states are zero for all T in R plus ok it doesn't matter what T is if I put zero states it is almost like saying that at equilibrium if my right hand side of the system is zero then this function which I am trying to analyze use to analyze the system cannot be non zero when the states are zero ok doesn't make sense so it has to be zero when the states are zero there has to exist a class K function Phi such that this function dominates Phi norm of X notice how the class K function has been used ok I mentioned very clearly that the class K functions have domain and range as non negative reals ok so but we are talking about states so how do we compare? we use the norm of the state as the argument ok the argument of a class K function will always be the norm of the state it can be a weighted norm no problem yeah but it has to be a norm ok alright so why because the norm is always non negative so by virtue of taking the norm I made the argument non negative ok alright great so important thing to remember is that this does not mean that V is strictly increasing V only has to dominate a strictly increasing function notice this look at this picture V doesn't have to be strictly increasing itself if I think of Phi X as this function strictly increasing continuous function which is zero at zero and V itself is also zero at zero V does not have to be strictly increasing it can oscillate but it has to remain above this line also remember my states were always required to be in this ball bounded ball therefore this domination also does not need to last forever it only needs to last until the norm of the states in fact this is not X but this is norm of X yeah this only has to last until the norm is less than R beyond that it doesn't need to last ok so two things this characterization number two does not mean that V is strictly increasing only requires to be bounded by a strictly increasing function and does not have to be bounded for all states yeah it has to be bounded only for certain states in a certain ball of radius R ok that's how we are right doing this characterization and that's it this is what is positive definite function this is how we define it ok we have still not connected it with the matrix definition for linear system we will go there ok so of course like I said positive definiteness will connect to asymptotic stability we look at how don't worry about it now if you look at if you if you think of this norm X and then you know norm X is something like this this is the two norm for example ok then a phi norm X is basically some and if you can think of phi norm X as something like this this is a class K function remember I hope you are convinced is a class K function yeah I'm sort of using this I mean going reverse and constructing a V a positive definite V by the way usually that's not the case first you are given a V then you have to think of a phi I'm sort of going the opposite side just to give an example ok you know the two norm the Euclidean distance and then suppose I construct this class K function this is a class K function I hope you are convinced is anybody not convinced that this is a class K function all you have to verify is that it's 0 at 0 it is it is strictly increasing do you believe it strictly increasing this can be written as 1 minus 1 over norm X square plus 1 it's strictly increasing and it has to be continuous or it is continuous in the norm yeah not difficult to see that it's continuous in the norm yeah only issue would have happened here but there is a square here obviously not an issue ok so this is continuously increasing 0 at 0 valid class K function so if my V T X is something like this this day yeah then this is going to dominate phi norm X yeah for all T greater than T in R plus ok for all T in R plus this is going to dominate this guy so this V is a positive definite function ok notice how different it is from your linear system sort of positive definite function characterization is rather different ok I hope that's clear. Thank you.