Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 2 : Lecture 12 : Stability- Part 5

Great, so we specified this condition for stability for linear systems which is connected to the state transition matrix and let's see how to prove it. We are actually saying this is equivalent, alright. So like I said it is actually in a lot of linear systems books you will not see epsilon delta definitions but you will see this as the definition for internal stability, alright. So of course let's see how it is equivalent. So okay, unfortunately I repeated it, I did not need to. The solution looks like this.

So this is the how you define or write the solution for a linear system in terms of the state transition matrix, right and you are already given initial condition x0. So let's start with assuming that this is true. So if this is true we want to prove that the system is stable in the epsilon delta sense that we just defined, okay. So if the RHS holds I know that the norm of x is less than equal to norm of this guy, right.

So in fact, I don't know what happened, okay, right. So I have sort of skipped a step. So norm of x is actually equal to, I don't need to say, norm of x is actually equal to norm of the state transition matrix multiplied by the initial condition vector which by this property is less than equal to norm of phi t t0 times the norm of x0, right. Just by using my induced norm inequality, alright, not doing anything fancy here. And I have already assumed that the right hand side is true so I have an upper bound on this.

So I get this guy, alright, excellent. Now if I am given an epsilon I choose my delta as this, yeah. Delta is just epsilon divided by this guy, yeah. Because it's obvious that if this happens then norm of x is less than equal to k t0 epsilon by k t0. I am done, right.

Well in fact this is becomes less than, not less than equal to less than because remember that my initial condition is strictly less than delta, okay. So keep these in mind, the less thans and less than equal tos. In the stability definition everything is strictly less than, yeah or strictly greater than. So delta is strictly positive, epsilon is strictly positive. Initial condition x0 is strictly less than delta then all trajectories are strictly less than epsilon, okay.

So x0 is strictly less than delta therefore this is not a less than equal to but a strictly less than. Just keep track of these. These are sort of important, okay. I am not going to discuss too much in length on why but we like to work with open balls or open sets, okay. And the set norm x t less than epsilon is an open set, yeah.

But norm, so this is an open set but if I take this guy, this is a closed set, okay. We do not like working with closed sets. We do not like them. Basically we do not like to work, I mean even when you are doing geometric control and so on, we will see we do not like working with manifolds or any spaces with boundaries. So as soon as you have something like this, there is a boundary here.

It impacts differentiability and so on and so forth. What happens on the boundary and these are annoying things we do not like to consider so much, yeah. So we like to work with open sets because there is no actually boundary here. It goes all the way, I mean very close to epsilon. So yeah, we are fine.

So keep these in mind just as a I would say just to be a little bit more precise, okay. It is good to be precise sometimes, okay. But the important thing to remember is that very easy to choose a delta given an epsilon. I mean I think we have done enough examples for you to get a feel for this I hope. You just write the solution, okay.

And you write the solution here and you make an inequality on the solution if you want to and then what do you need? You just need, in fact how did I get the delta choice? I needed this to be less than epsilon. I need norm x to be less than epsilon. From this I can in fact directly get what I need my x0 to be smaller than, right? Because x0 has to be smaller than epsilon over kT0 for this to happen, right? Okay. So I have simply used these inequalities. I need x to be less than epsilon.

So if I take this quantity which is possibly larger than norm x and I make that less than epsilon then it is guaranteed that x is also less than epsilon. So I have just used these inequalities smartly to my advantage, right? So this is how I always find a delta given an epsilon. So if you get a problem on stability this is what you have to do, yeah? You write the solution and if you have an upper bound on the solution or the solution itself just upper bound it by epsilon and you try to find what is the initial x0 because the solution will always contain the norm x0 itself, okay? Always. Without that, without initial condition there will be no solution. The only thing is in the nice linear case initial condition appears linearly, yeah? This is one of the outcomes of linearity, right? Which will not happen in a nonlinear case, okay? You will not necessarily have linearity in initial conditions either, okay? Okay, very good.

One side too easy, no problem. Other way round, if I assume stability holds and I want to prove this, if I assume this system is stable and I want to prove this happens then we have to make some interesting moves, okay? If LHS holds it implies what? If I am given an epsilon, let's be precise, if I am given an epsilon which is positive there exists a delta which potentially depends on initial condition, initial time and also epsilon but okay whatever and is also positive, right? Such that if my initial condition lie in a delta ball then my solution lie in an epsilon ball, okay? This is exactly the definition copied, yeah? Now I say something interesting. I say that I will fix a TA and choose an XA such that this happens. What is this by the way? What is the left hand side? What is the right hand side? What is the left hand side? This guy? What is this? Is it? Yeah, it's weird, no? It is not, it's not, first of all I did not say XTA equal to XA or anything like that. Notice I did not say that, yeah? Okay I did not say that.

So this is nothing, it's not the solution at time TA or T0 or anything like that, okay? It is just the product of the state transition matrix times some vector XA, okay? What is exactly happening here? The first thing I did is I fixed a time, right? So that this matrix now becomes a constant matrix, right? Once I fix a time a constant matrix. Then what is my sort of a claim, this is actually a claim in a sense, right? I am saying I choose an XA such that the norm of this matrix, yeah? What would be the norm of this matrix? By definition what would it be? Norm of this guy divided by this guy over all possible X, yeah? But I am saying and I had even made a claim here, right? That is always greater than equal to. So norm of A times X is always less than equal to norm of A times norm of X, right? But I am saying there exists an XA such that this equality holds, okay? In general if you plug in arbitrary X this is true, yes? Just by definition of the norm, the induced norm. But I am claiming that there exists an XA because I am now talking about a constant matrix phi tA t0 whatever A, okay? I am claiming that I can choose an XA such that this I get an equality here. Why do you think I can do that? Really I took a supremum.

You remember the supremum, right? The supremum is like, you know, like least upper bound. It does not have to be in the set and all that. It is the supremum, I mean we saw these examples, right? 1 minus e minus x and where it is and then you are talking about the set which is, so 1 minus, sorry, 1 minus e minus x is what it is? No, that was not 1 minus e minus x, right? It cannot be. 1 plus e minus x, we do 1 plus e minus x, right? 1 plus e to the power minus x, yeah? No, does it work? No, no, no, no. How did we choose it? 1 minus e minus x and so the set was basically this guy.

I get everything from 0, 1, right? But the supremum is exactly 1, right? Not in the set and so on. Why do you think this does not happen in this case? How can I get an exact equality here? You are saying that is what will give you the equality. So, what I am saying here is that of the irt nearer this is more of co-aff lipstick. So of the right side, is not it? So, we show the universe scream. See, just some of you off- SHE Members.

Semi lightensDelegation. Step1. Step2. So, this is easier if you took the dot at the Explosions section. Yeah, in this case you are talking about all of Rm, ok, which is both open and closed, right. You have all the nice properties that you want in all of Rm, ok.

The second thing to sort of should help you convince, should help convince you is that I have formulae here for norm of A which is independent of X, right. I mean anyway its supremum is expected to be independent of X, yeah. But I have some formulae which exactly gives me what my norm is, ok. So the basically again not a proof, this is not a proof.

If you ask me for a proof, I will have to hunt for a proof in the sense that it will have to be, it has to be based on, it is basically based on the idea that the reals have this nice Banach space type property, ok.

So if I think all of Rn, it has a Banach space, it is a Banach space, Hilbert space, whatever. It has all the good properties which we talked about, ok. So it is essentially based on the fact that you are taking all of Rn, you are not making any funny sets and it is a Hilbert space or a Banach space, ok. So that is why you will always have an XA for which given any constant matrix you will be able to find that equality. So basically the max and the sup will become the same, ok.

That is what we are saying, ok. So that is basically the idea and that is what you rely on to prove this, ok. Once you have such an XA which gives you this equality, ok. Here I have just said that it exists by the definition of induced norm but it is not as simple as that. So little bit more than that just like we said, yeah. Once you have such an XA, what is the good thing? You can actually now play with this system, ok.

What do we do? We consider this sort of an initial condition, ok. Do not worry about how this is going. It will sort of, you will close the loop and see how things worked out well for you, ok. But this is the clinching thing here, yeah. Once you have such an XA, I construct an initial condition, ok.

I construct an initial condition which is this, ok. What does this give me? If I take a norm, it gives me delta by 2 and these cancel out, right. So I know that the initial condition is bounded by delta, right, because it is equal to delta T0 by 2, therefore it is upper bounded by delta. Is that ok? I have just constructed this X0 in this funny way, ok. I am basically going to try to use this definition to get to this sort of an inequality, ok.

So I am going to, I am basically trying to use elements of this definition. So I have constructed my initial condition using the delta that I got from stability, ok. So I know that this, the way I have constructed, I know that norm X0 is less than delta, which means that norm XT corresponding to this X0 will be less than epsilon, right. So norm of XTA, I don't compute XT for arbitrary T, I now compute XTA, ok, which is phi TAT0 times X0, ok. Phi TAT0 times X0, I have chosen this X0 in this interesting way, ok.

Again this is a scalar, but anyway this product, the norm of this product is less than epsilon by my stability assumption, right. So this is less than epsilon by my assumption of stability. So this is a scalar, goes out, ok, and this product I have already claimed is actually equal to this, yeah. Norm of phi times XA is actually equal to norm of phi times norm of XN, because I have chosen this XA in this very special way, alright, ok.

And this is less than epsilon. You can see that I am already close to the end now, ok, not difficult now, because I have the norm of phi TAT0, basically I have the norm of phi, which I want to bound, right. So I am going to get a bound of norm of phi here, right. So that's essentially what I have. Again I have repeated it. And from here I get norm of phi, these XA's cancel out, that's the nice thing.

XA plays no role anymore. And I get the norm bound as this guy, which is some KT0, ok. Now you might say that I took a particular TA and I mean I took a TA and so on, but remember I said fix TA to begin with. So if you say that I fix TA, you only prove for one particular TA, I will say that you fix some other TA or a TA prime, but you can do the same arguments again and you will get the same inequality again. In fact nothing will change, it will be exactly the same, because the right hand side does not contain TA or XA or anything like that. All the, everything that we introduced goes missing from here on the right hand side.

Therefore you can keep changing this TA to TA prime, TA double prime, triple prime, whatever, different choices of TA, right hand side is not going to change, which means that for arbitrary choice of T this has to hold, ok. So basically you proved the other side of the argument also, ok, make sense? A little bit involved, but the only thing that is important here is the existence of an XA such that this happens, ok, alright. All of this works out again because Rn is a very very nice vector space, alright. If you don't have very nice vector spaces, but we don't work with the non-nice ones, again let me be honest, yeah, because we have already said that we are working with some non-linear space, inner product linear space where you have Cauchy convergence is equal to convergence, so obviously we are already sitting in some very nice vector space, ok. So having this kind of a property is actually not so unusual, ok.

So what about uniform stability? I mean nothing will change, you will get the same kind of result, ok. One side this to this is anyway too simple because your, if a uniform stability this K will be independent of T0, right, that's how you will have uniform stability because you sort of remove the dependence on initial time. So therefore this will, there will be no longer a T0, it will be a just a constant K, ok, just a constant K for all T0, alright. And once you have that going from here to here is very easy because K is independent of T0, so delta is independent of T0, done. On the other side also if you see no longer dependent on T0, right, because you assumed uniform stability, so the delta, so this delta is also independent of T0, you started with uniform stability, so obviously this has no T0 here.

Once you don't have T0 your x0 does not have T0, ok. And this guy doesn't have T0, alright, here also there is no T0. So essentially too simple, right, this T0 dependence vanishes here, ok. So again you get a K which is independent of T0, so it works out on both sides, yeah, so very simple which is why I am not giving a separate proof but all you have to do is remove the T0s from your proofs, that's it, that's all you do here, alright. Great, finally for linear systems asymptotic stability is actually equal to stability plus this sort of a convergence, ok, so attractivity is this guy but this is pretty evident, right, because if you write the solution you know that as your solution, as time increases this goes to 0 therefore whatever be the initial condition your solutions will converge to 0, right.

So this is essentially attractivity, in fact global attractivity but I already said that local-global is irrelevant in this context, ok. So if this goes to 0 then initial condition is irrelevant, it is just some scaling constant, ok, so everything goes to 0, alright. If there are no questions we will sort of conclude here, yeah, so this is basically what we have for stability and I believe from next time we will be able to start talking about the Lyapunov theorems, alright, so already we will get to the crux of how to analyse stability for non-linear system without actually solving the system, as you can see very hard, yeah, even these conditions phi norm of phi less than equal to k T 0 or k virtually impossible to you know claim anything on without actually solving the system, so you know, so this is something you have to do, you will have to do the Lyapunov theorem without which for non-linear systems you can't claim anything, yeah, except with the linearization methods which are restrictive, yeah, because they don't give you a basin of attraction, alright, so we will start with those from next session, ok. Thank you.