Nonlinear Control Design

Prof. Srikant Sukumar

Systems and Control Engineering

Indian Institute of Technology Bombay

Week 2 : Lecture 11 : Stability- Part 4

Alright, so great, so we have pretty much all the stability notions that we require covered yeah pretty quick yeah. Once you understand like I said stability, uniform stability, you are more or less done. All the other notions are just you know sort of combinations of one or the other of the two yeah and you sort of understand how to strengthen them yeah. You don't have to necessarily remember the sequence and all that. Once if the first strengthening is imparting uniformity which is independence with respect to initial time and then the second strengthening is sort of you know giving some kind of global property which is basically removing bounds on initial conditions ok. So initial x0 if it is unbounded that is you can allow for any initial condition then you have global properties alright, simple great.

And this is very critical for non-linear systems of course. For linear system this is irrelevant global properties and local properties are all the same yeah. There is no difference between local and global for linear systems. So that is obviously there.

If you are talking about time varying linear systems you still have to talk about uniformity ok. So again the other thing to also remember is that if your system right hand side that is f yeah we talk about systems of the form x dot equal to f t x right. If this f here is independent of time there is no time explicitly appearing on the right hand side then the system is uniform for free. There is no uniformity to be considered separately ok. So these few things always remember linear systems local global no difference.

If the system right hand side function that is vector field has no explicit time argument. Why do I keep saying explicit because the states themselves are functions of time when you solve them and all that. But if time is not explicitly appearing in these expressions then the system is time invariant or autonomous yeah. These words are used exchangeably for continuous systems yeah. So then uniformity is free you do not have to evaluate separately.

If you obtain global asymptotic stability you have obtained global uniform asymptotic stability ok alright excellent. So just keep these in mind. Now we sort of go to these examples right these interesting examples that sort of give us different funny cases if you may yeah. So this is exactly the question that you asked the system which is attractive but not stable yeah. This is the system yeah unfortunately you cannot solve this by hand at all yeah and no analytical solution.

But the phase plane plots looks like this ok. So this is x1 dot this is x2 dot this is the system I mean you can think of it as some kind of a constructed system I guess alright. But well it exists so yeah. So the phase plane plot like I said is just a plot between the two states in this case x1 on one axis and x2 on the other axis. So there is a sort of a bifurcation in this system that is inside this blue sort of a you know aerofoil type shape all the system trajectories look like this ok.

So how do you make the phase plane plot is very standard in MATLAB, Python and so on. You basically give a bunch of initial conditions and you look at how those initial conditions evolve. So in this case if you start a initial condition here it will evolve along this like this. So you get a idea of what is the phase plane curve. So in fact in MATLAB you have standard functions which will directly make phase plane by automatically randomly initiate initializing and then they will just directly make the phase plane plot for you for several cases.

So the basic idea is this system is such that any initial condition inside this sort of bifurcation will have this kind of a shape ok. They will all look like this and if your initial conditions are outside then the trajectories look like this ok. So if you start outside trajectories look like this ok. So the interesting thing to see is it is attractive right because it is wherever you start you are always getting to the origin right. If you start inside outside it does not matter it is going to just do this and go or it is going to do this and go it does not matter it is going to the origin wherever you start ok.

So attractive in fact you can even claim global attractivity in this case yeah. I do not know about uniform well I guess it is uniform for free no time argument on the right hand side right. So it is uniform in fact it is globally uniformly attractive and probably the best attractivity property you can imagine right but it is not stable why? Absolutely look at this if you give me any epsilon ball ok any epsilon ball around the equilibrium origin in this case right. You make an epsilon ball like this I have actually made the ball very specifically for that you make the epsilon ball yeah ok. Then you cannot remain in this epsilon ball for any delta ball why? Because any delta ball will definitely have initial conditions outside this guy right.

If your initial conditions well actually it does not even matter it is outside actually it does matter if it is outside or inside yeah. There is no initial condition which will remain inside the epsilon ball ok. They will always go out like this yeah if epsilon is like this. Now if I of course if I make a very large epsilon let us see let us see that case also alright great I made this epsilon ball now ok now what does this work? Definitely ok unclear if this works or not but there is a hope that it might work because if I take I mean but potentially I could take really small delta like this. So, that the trajectory that is see the inside trajectories are not a problem this sector trajectory is not a problem because they will remain inside this wing and I deliberately made epsilon larger right.

But any trajectory starting here has to go around this. Now I do not know if it will remain inside this or not but the point is if I make this delta really tiny I am sure it is going to hug this very closely and then not escape this circle ok. So, there is a possibility of finding a delta but that is not the stability question right. The stability challenge is that you give me any epsilon and I can give you a delta yeah. So, for large epsilon yeah seems like great working but for small epsilon no ok.

So, it so then it is that is it I do not have to do anything further if it does not work for even one particular choice of epsilon that you give me system is not stable ok. The system is not stable but it is attractive ok alright. Then there are the simple nice examples like this pendulum at the non-linear pendulum actual pendulum it has a equation that looks like this right. And I do not have to actually do anything to prove that this is asymptotically stable you can just I mean I can prove it not by solving it difficult to solve the system also just because of the sinusoid you will not be able to analytically solve it yeah. But you can just look at the movement of a pendulum you know that it is attractive yeah because of this damping term this damping will come usually from the friction on friction in the pivot point.

So, if you make any pendulum move it is going to stop at the bottom right. So, it is attractive yeah the bottom position that is theta equal to 0 is an attractive equilibrium right. And stability is actually rather easy in this case yeah in the sense that I can guess it yeah, but again I cannot analytically solve it yeah although this system looks so simple still I will not be able to analytically solve it to get stability. How I would get stability is by linearizing yeah I would linearize this, this will become x2 dot is minus x1 minus kx2 around x1 close to x1 equal to 0 near x1 equal to 0 this becomes x1 dot is x2, x2 dot is minus x1 minus kx2 ok. So, that is actually near x1 equal to 0 if I linearize I will get x1 dot is x2 and well x1 equal to 0 and x2 equal to 0 and x2 dot is minus x1 minus kx2 and of course k is positive.

So, this is a stable system right this is in fact asymptotically stable system but it is definitely stable ok. So, this is one of the methods that I do not discuss in this course, but it is possible to get or obtain local properties of a non-linear system by linearizing the non-linear system and obtaining the linear system and looking at its corresponding stability. So, if you linearize a non-linear system and the linearization is stable then non-linear system is locally stable. Linearization is asymptotically stable non-linear system is locally asymptotically stable yeah. Why I do not discuss these this particular method is because this method does not tell you how local is local.

It just tells you that if the linearization is asymptotically stable non-linear system is locally asymptotically stable. It does not tell you how local it does not tell you the delta in those definitions ok and a lot of times you need to know those ok. I mean if you are especially wanting to operate such systems you actually want the value of that the basin of attraction that it is called yeah. So, Kahlil uses this word a lot basin of attraction basically it is the delta in the attractivity definition ok. So, we need to know that in a lot of cases which is why I do not talk about that method here, but anyway for this particular illustration it is ok.

So, this is an asymptotically stable system yeah local yeah not global ok. Not global because the inverted there is an inverted position that is x1 equal to pi which is also an equilibrium 0 0 is an equilibrium pi 0 is also an equilibrium right for this system ok. So, the pi 0 equilibrium is unstable ok. We are not talking about but so if you are talking about stability of this equilibrium it is only local because there is another equilibrium ok. So, whenever a system has multiple equilibria you cannot claim global properties ok.

It is just like optimization right. So, well in optimization you still can right because you say if you have multiple minima you just find the least minima in the entire domain and you say that is the global optimizer right. Here you do not have any such thing. Here if you have multiple equilibria then you cannot say that any of them is global because an equilibria itself means that you never move from there. So, this inverted position if you start here and there is no disturbance and obviously idealized system.

If it start here and there is no disturbance then this is a trajectory which does not converge to the 0 0 point yeah. There is actually a trajectory right because pi comma 0 is an equilibrium. So, if I start at this pi comma 0 I remain at pi comma 0 because the right hand side becomes 0 right. So, I do not move from there. So, this is a trajectory for all time I remain here yeah.

So, my property cannot be global right because if I start right here I do not converge here and global requires that for every initial condition I should be able to reach the equilibrium right. In this case that is not happening. So, whenever you have multiple equilibria it is virtually impossible to say that it is a whatever you have achieved is a global property ok alright make sense. So, what most people do is they try to say that ok there are some you have this equilibria, but it is an unstable equilibria therefore in reality you will never actually stay here and so on ok which is true right. I mean if you take a pendulum even if you start here you will never stay here right I mean however much your damping is even if it is slightly slightly off from this which it will be in the real world you will go back to the 0 0 equilibrium ok, but it is not global ok sadly ok.

So, remember that not global property alright this system is globally asymptotically stable for sigma positive very easy just solve it yeah this is easy to solve this is what you have you get this this is the solution yeah believe me this is the solution yeah I checked ok. So, this you can see that as t goes to infinity what happens this term is going to go to infinity. So, this entire thing is going to go to 0. So, attractivity is guaranteed I have to prove stability, but yeah I mean yeah it is stable I guess I have asked that as a example ok yeah nice that is an exercise to prove stability yeah because attractivity is obvious right I mean you do not have to do any work at all yeah, but stability you might have to do a little little little bit of work ok. So, that is an exercise here you have to prove that this is globally asymptotically stable.

Again I say only globally asymptotically stable because uniformity is free right hand side is independent of time ok. So, no need to evaluate uniform if I say GAS it is GUAS alright ok good. So, this is a of course I sort of wanted to give an example of non-uniform asymptotic stability also alright. So, you can have such examples also. So, all I do is I sort of introduce a function of time here yeah if this was 1 if this was 1 just think about it this is just a linear damped oscillator right.

So, spring mass damper if this was 1. So, but all I did was I introduced some time varying quantity here and it so happens that the solutions look like this I mean. So, I mean it turns out that it is non-uniform asymptotic stable ok. So, basically you do not have uniformity in this case. In a lot of these cases it is not very easy as you can see to solve them and so on. If you look at this, this is just a linear system by the way yeah linear time varying system.

So, as soon as you go from a linear time invariant system to a linear time varying system your life becomes pretty complicated already yeah you do not even have to go to non-linear. I mean what I would say is linear time varying systems are sometimes harder to analyze than non-linear systems non-linear autonomous systems ok. So, does not make your life significantly simpler or anything yeah. So, even to evaluate a non-uniformity using these definitions would be rather difficult rather difficult ok. Any questions? If you have a single unique equilibrium point for a non-linear system your question is does asymptotic stability guarantee global asymptotic stability? I would say no, no, no does not.

You can have a case where you do not converge to that equilibrium also far away from the origin you do not get to the origin yeah. I do not think this can be said in general that if you have asymptotic stability of global asymptotic stability for a non-linear system just because you have one equilibrium ok. If you ask me for such an example I will struggle a little bit yeah. I will have to think ok I will have to think. I think a version of this van der pol oscillator might be such an example yeah I have to sort of think about it yeah.

There may be a version of a van der pol oscillator which will which sort of does exactly what I am saying that it does not do it is almost like this in fact if you look at this system yeah well this is still nice inside and outside in both places it is converging to the origin, but you might have some kind of bifurcation which beyond which it does not converge to the origin, but inside it does yeah. So, non-linear systems can exhibit a very very wide variety of behavior. So I cannot claim that in general if you say asymptotic stability then essentially we are saying asymptotic stability implies global asymptotic stability more or less right. No, I would not have any reason to make a separate definition. No escape velocity is slightly different because I see you are talking about escape velocity in terms of launches like in terms of launching and stuff.

Yeah I understand what you are saying. Maybe yeah maybe that kind of a model yes if you think of a rocket that you are trying to write equations for and then you have gravity acting on it so there is an escape velocity if it is so basically if one of the states is too large then you do not come back and fall into the ground, but if your velocity is smaller than this particular number so one of the states is small then you do come back yeah yeah maybe if I model it appropriately it will probably exhibit this behavior. So the model I mean I am wondering the model also has to exhibit this behavior physically yes absolutely yeah physically yes so yeah I mean you can always I mean if you can think of it then you can model it so I guess you can have such an example also yeah. So non-linear systems very very wide variety of behaviors you cannot say that in general ok. Alright any other questions? Ok good this is one of the simplest examples of course right this is the global exponential stability this is a linear system right like I said linear system linear time invariant system especially it is like exponential stability is sorry any stability is exponential stability ok nothing less than that speed whatever very fast, but again linear time varying systems that may not be the case so it is a very fragile sort of a thing which works only for linear systems and not more alright. So like I said for linear systems in general uniform asymptotic stability and exponential stability are also the same are the same yeah so there is no difference really.

In fact how do you evaluate it you just look at the matrix eigenvalues and so on and so forth yeah I mean pretty simple all of you understand how to do it you can also do the Lyapunov tests and all this usual things that we have been doing, but you do not have to but we will look at those conditions yeah in order to look at those conditions we have sort of redefined these matrix norm notions, but I am not going to repeat them I believe you already know them yeah just the notion of the induced norm repeated here then how to evaluate some induced norms again repeated here just the formulae and then these inequalities which are yeah which are just the induced norm inequality right this inequality is coming from the induced norm definition itself right because induced norm is the supremum yeah therefore, A x divided norm A x divided by norm x is always greater than norm A ok because it is the supremum right. So for all values of x norm A x divided by norm x has to be greater than equal to norm of A I hope that is evident right because sup is basically a extension of the maximum right and that is how you get this ok. Did I say this correctly I said it the other way around sorry from here you have norm of A is always greater than equal to norm A x divided by norm x yeah just by virtue of the fact that I took a supremum here from here I get this sorry I said it the other way around yeah. So just the norm inequality induced norm inequality and this is the Cauchy Schwartz like inequality like that we sort of proved for general vector spaces last time ok. So when do we say that ok this is a little bit of proof so anyway that is fine.

So one of the conditions for stability is this guy for linear system so this is a general time varying linear system ok with some initial condition because I have time dependence so therefore my initial condition is also at a particular initial time not necessarily 0 alright. So the condition for stability is something like this ok basically what is this notation I hope all of you know this notation this phi capital phi by the way is the state transition matrix yeah or the fundamental matrix whatever you want to say yeah and you also know that the solution at any time can be written as in this way yeah this is how you use the state transition matrix in fact the state transition matrix describes the flow for the linear system yeah because it tells you map the if you fix the t it tells you how you map initial conditions right different initial conditions I can take different initial condition I keep multiplying with phi I get a flow how my initial conditions are getting mapped ok. This is just a linear equivalent of whatever the flow in fact that is why you see the similar notation getting used there it was a small phi here it is a capital phi yeah so what we say is the condition for stability is that the norm of this matrix is bounded by some function of initial time ok. Thank you.