Nonlinear Control Design

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Week 2 : Lecture 10 : Stability-Part 3

Okay, welcome to control of non-linear dynamical systems. We are already in the second week, right? And we have already started some more serious material. So last time we started looking at stability, okay. And how we started talking about stability was of course first by fixing a system initial conditions, defining what is the solution. We also spoke a little bit just to spark our interest on the notion of the flow, yeah, which essentially sort of tells you how a bunch of initial conditions map to some final conditions after a certain amount of time, okay. So beyond that we started talking about the notion of equilibrium, alright, and how we sort of care about isolated equilibrium, yeah.

And we of course gave an example of non-isolated equilibrium, very easy to construct. Yeah, I also showed you this pen which is rolling on a surface, yeah. It is completely non-isolated equilibrium because every point is an equilibrium. Without disturbance you do not move from any point at all, okay.

So once we have the notion of equilibrium, all notions of Lyapunov stability are defined with respect to an equilibrium, alright. No equilibrium, no notions of stability, okay. So this is how Lyapunov stability works, yeah. There is of course Lagrange's notions which we do not really talk about but there you do not particularly need notions of equilibrium there. You are talking about boundedness, ultimate boundedness, uniform boundedness and things like that, okay.

So slightly different notions which do not particularly talk about stability in the sense of Lyapunov. Here we need an equilibrium. So we started with the definition of stability. So obviously you start seeing the equilibrium point appearing everywhere here, yeah. And stability essentially is an epsilon delta definition the way we understand it, yeah.

It just codifies the fact that if you start close to the initial condition, sorry, if you start close to the equilibrium, you are expected to remain close to the equilibrium for all time, alright. And that is codified as given an epsilon, there is a delta such that if you have, if you start within a delta ball, you remain within an epsilon ball, okay. So we of course understood this a little bit better hopefully, yeah. We sort of tried to compare it with boundedness and we sort of understood that it does not really compare with uniform boundedness well at all, yeah. One does not imply the other.

Then we went on to talk about uniform stability which is just t0 removed, okay. So the delta does not depend on the initial time anymore. And we clearly said that whenever we talk about uniformity in this course, we are always, you know, talking about uniformity with respect to time, okay. So whenever we say uniformity, time is involved, okay. So that is the idea, alright.

And then we of course looked at this very very nice example. Why is it a nice example? Because we can actually construct a solution here, yeah, and we can look at some interesting properties and we in fact saw that this system is stable but not uniformly stable, okay. So we will of course explore more properties of this system as we go along and make more definitions, alright. So we will explore further properties of what this is but for now we know that this system that we have looked at is stable but not uniformly stable, okay. So what we do today is we continue to talk about more properties and we move towards asymptotic convergence or asymptotic stability, okay.

So that is the idea for today's lecture at least, yeah. So we start by assuming that the equilibrium is the origin. Yes, please. So the equilibrium is simply any point in the state space where from which you never move under ideal circumstances, okay. That is you do not deviate from an equilibrium unless there is a disturbance present or anything like that, okay.

So the equilibrium is simply a point on the state space from which there is no movement, okay. The states never move from there, okay. And how we compute it is pretty straightforward. We just equate the right hand side to zero. That is it, okay.

That is how we compute it. That is how we did it for this example also, alright, okay. Great. So for the rest of the presentation we assume that the equilibrium is in fact the origin, yeah. So it is not difficult to shift the origin so that you can ensure that your equilibrium is always the origin.

You just do a simple change of coordinates like this. Yeah, we are very used to doing this whenever we are talking about the tracking problem for example, right. Whenever we are doing tracking we always do some kind of a subtraction to make sure that we are always talking about going to the origin, okay. This is what we are comfortable with. It is as simple as that, yeah.

Also makes our notation simpler. I don't have to keep writing X, E everywhere, alright. So from now on assume that the equilibrium is the origin, okay, 0, 0 in the state space, alright. Great. So we talk about the notion of attractivity now, okay, because this is the next important notion, alright.

So what is attractivity? For all t0 there exists a delta. Again possibly depending on t0 such that if you are within a delta ball of the origin or the equilibrium in this case as you can see then as t goes to infinity you approach the origin, okay. So this is simply attractivity the way you would understand it. The only difference is you can see that it is defined locally, yeah. It is a local definition because it is saying that if you give me a initial time or t0 I will give you a ball of certain radius within which you have to start if you want to get here, okay.

If you start beyond it we cannot guarantee anything, okay. So that is the important thing to remember. The notion is local because I start within a delta ball. I have to start within a delta ball.

That is it, okay. Then obviously we try to strengthen these notions, right. There is the notion of uniform attractivity. Remember that I said that whenever we talk about uniformity it is always with respect to time. So the only thing that depends on initial time here is this, right. So for uniform attractivity this delta is independent of t0.

That is it, okay. As simple as that. So very similar to stability, uniform stability. The delta was depending on t0 and then it does not depend on t0. So you can see the same thing happening here, okay. I hope that is clear, alright.

Make sense? Okay. Then specialize further. So attractive strengthened to uniform attractive, further strengthened to globally uniformly attractive. What is that? The delta goes away completely. You can start at any initial condition, any initial time and you approach the origin as t goes to infinity, okay. So you are strengthening the definitions as you go down here, okay.

So remember that when we spoke about stability one of you even asked I think that is there, is it local or global? Stability has no notion of local or global. Stability is just stability. If you notice there is no, if you give me an epsilon I give you a delta, okay. It is not local or global there.

There is nothing local global. But here there is, okay. Very clearly. Convergence is local or global. So it is or attractivity in this case. It is a parallel notion to convergence for series.

So attractivity is local or it is global, okay. That is it. Stability is a, there is no local global there. So remember this, okay. So we have strengthened sufficiently I guess.

Now the rest of the definitions are very simple. It is just a combination of these two properties, right. What is the combination? First we again start with the weakest notion. Asymptotic stability, okay.

Acronym A.S. Alright. We use these acronyms a lot of times because they are very long sentences to say. So asymptotic stability requires a combination of stability and attractivity, okay. That is it. You already know what is stability.

You already know what is attractivity. If you have both properties for a system, it is asymptotically stable, okay. See I no longer require any more epsilon delta definition. In books of course you can, if you go to Vidya Sagar and you go to some other texts, they will probably formally tell you the definition of each of these. So then it is not required, okay. You have already defined stability, you have already defined attractivity.

If you have both the properties, then it is asymptotically stable, okay. And unfortunately different books have slightly different definitions, yeah. I would stick to what we are talking about here, okay. In most cases they are identical, okay. You can prove that one implies the other and so on and so forth, alright.

So don't worry about the slight differences. For example, if you look at Khalil, you might find a slightly different definition. If you look at the Vidya Sagar book, you might see a slightly different definition, yeah. Let that not, you know, sort of worry you.

One typically implies the other, alright. Then we have uniform asymptotic stability. Here I just qualify each property with uniformity here, okay. So I need uniform stability and I need uniform attractivity, okay. So this is just uniform asymptotic stability, alright.

So this is a pretty strong property, yeah. In fact, one of the strongest properties you can have for non-linear system. More often than not this is where you stop, yeah. Then neither of these or none of these conditions actually talk about any rate of convergence, alright. You, you can never, in fact in most non-linear systems you cannot actually say how fast you are going towards the origin, okay. It may be linear, it may be sub-linear, it may be logarithmic, whatever.

It could be very slow, okay. So you cannot actually guarantee but in some cases where you can, you can define the notion of exponential stability. Why exponential and nothing else? Exponential is the holy grail because linear systems give you exponential stability, right. Any linear system if you say it is stable, it is exponentially stable. It is nothing less, okay, alright.

Well, linear time invariant systems, alright. So what is exponential stability? There exists constants R, A and B positive such that this sort of a equation is followed, okay. Again vector norms, huh. Basically norm of xt is less than equal to A times norm of x0 times e to the power minus bt minus t0, okay. So this is the typical exponential decay and this is, this is to hold for all t t0 greater than equal to 0 and for all x0 less than R, okay. In fact you can probably write this slightly better and say that this is t greater than equal to t0 greater than equal to 0, yeah.

And for all initial conditions which are starting within a R ball, okay. So this is actually a local definition, right. Whenever you are requiring your initial conditions to start within some ball of some radius, okay, then it is a local condition, okay, because you are requiring initial conditions within some set, okay. Only then you converge is what you are saying here. If you start beyond that set you are not guaranteed anything.

All such properties are local property where your initial conditions are in fact required to start within some kind of a ball. You can again strengthen this to global uniform asymptotic, oh sorry, I apologize. This is actually a strengthening of this guy, yeah. So strengthening of this is the global uniform asymptotic stability.

Remember that there is no global local here. So this remains as it is, yeah. But this property there is possibility of a global counterpart. So you say that you require global uniform attractivity, okay. So all these other properties beyond stability, uniform stability and attractivity they are just a combination of these properties, okay, which is what makes things relatively easy in terms of at least writing the definitions, alright. So slightly off sequence I guess but anyway you had asymptotic stability, then you went to uniform asymptotic stability, then you went to global uniform asymptotic stability.

This is the strengthening, okay. Why the exponential stability in between is also makes sense, yeah, because exponential stability is also local, okay. So now I can move from exponential stability to global exponential stability. What would be the difference? This will go away, right. That is the only thing that is sort of keeping things local for you, right.

So this will go away. So that is what you see. There exists constants, now only two constants because R is no longer required, right. Only two constants here such that the same thing happens. Again I would say, yeah, T greater than equal to T0 greater than 0. Same thing happens but for all initial conditions now, okay. No longer requiring any restriction on the initial conditions and hence global, okay.

So what the exercise that is mentioned here is essentially to prove that exponential stability is stronger than UAS, okay. That is exponential stability implies uniform asymptotic stability. This is an exercise, yeah. And similarly global exponential stability implies global uniform asymptotic stability, okay. So you have to prove that this guy implies this guy and that this guy implies this guy.

Only one way, not the other way of course. They are not equivalent, okay. So what we are trying to say is that exponential stability gives you something more than uniform asymptotic stability. And similarly global exponential stability is something more than global uniform asymptotic stability. So that is what you have to prove, okay. So you have to start by assuming that you have this kind of a condition and then you have to prove uniform stability, you have to prove uniform attractivity, okay.

Alright. Okay. Any questions? Yes. Yes, if whenever global is not written, you will assume that we are talking about local, okay. Many books do write local, use the word local but more often than not we don't. We don't say local uniform asymptotic stability, local asymptotic stability and all that. We just say asymptotic stability. If the qualifier global does not appear, then you assume it is a local requirement, okay.

That is standard. Yes. Is there an attractive system that is not stable? We will see. So it is a good question. If there is an attractive system that is not stable, stable system not attractive. Stable system not attractive is very easy.

Any example of stable system not being attractive? Oscillator. Oscillator, yes, standard oscillator, spring mass damper, no damper, spring mass, standard spring mass system, oscillator, they are all you know non-attractive. They are stable, non-attractive, yeah. Linear oscillator, let's stick to linear oscillator. Non-linear oscillators, funny things might happen.

But what you are asking is the other way around. If there is an attractive system that is not stable. So we will see examples, we will see very interesting examples, yeah, alright. I will immediately went to the example. I want to go back first to this system, okay.

The system we considered, okay. Now let's look at this system and I really hope that you at least remember what happened here. So if you look at this system, we wrote the solution in this form, right, where gamma is of course a function of initial time, right, and this is the initial condition, yeah. But this is the basic evolution, okay. Can you tell me if this system has any of the, so you already know that it is stable and not uniformly stable, okay. Does it have any attractivity property? Is the system attractive or does it have any attractivity property? Because that is what we need to claim asymptotic stability or something like that, right.

But do you think it has any attractivity property, just looking at this solution? What does attractivity require? If you start with a norm less than delta, then you converge to the origin, okay. So if you start with norm less than delta, you converge to the origin. Do you think that is going to happen here with this system? Yes, why? T squared. T squared, absolutely. Once you fix the initial time, yeah, then because of T squared, because T squared, we discuss this, right.

In fact, this is the picture. Beyond a certain time, whatever is in the exponent, this guy, is going to become negative. And negative exponential means what? Dk, right. You are going to go to the origin. So exponential of negative quantity, in fact, fast decreasing negative quantity is going to go to the origin, okay.

Okay, great. What is delta in this case? What is the restriction on initial condition? I said norm x0 less than delta implies you converge to the origin as T goes to infinity, right. And you are right, as T goes to infinity, obviously converges to 0. What is the delta? What is the bound on initial condition that is required? M. M is somehow the bound on this guy, right.

So M is you are saying the supremum of this. But why? How do you, but it is on an exponential, right? First of all, you are taking exponential of this guy, first of all. What is f now? There is no f. I have an exponential of this guy, remember.

I do not have any general function here. It is a very specific function here. And M is the upper bound of this term here. So let us look at the definition again. You want this for attractivity, right.

Start in a delta ball. If you start in a delta ball, you converge to the origin. What is delta? You converge to infinity.

Absolutely. There is no delta. So trick question, alright. But there is no delta. You give me any initial condition. How does it matter? You give me any x T 0. The exponential, this exponential is once you fix T 0, so gamma is fixed. This exponential is always going to push me to 0 irrespective of what my initial condition was.

You can give me x T 0 as 10 to the power 10, irrelevant. This exponential is definitely going to, it is a constant. Whatever it is, it is a constant. This is also a constant.

So both of these are just some constants. Even if they are huge, it is irrelevant, right. Because this exponential is going to go down really fast as T goes to infinity. It is going to go to 0. So it is going to get rid of whatever these guys are. If you get 10 to the power 10 here, it will become less than 10 to the power minus 10 after a certain time.

You can always find that time also, alright. So in fact this is, then what? Globally attractive. Okay, great. So it has global attractivity.

Okay, excellent. What about global uniform attractivity? Yes, no. Because there is no delta. So uniformity only require delta to be independent of T 0 and all that. But there is no delta requirement at all. So it is globally uniformly attractive. Okay.

So what property does this system have now? It has stability and global uniform attractivity. So what does the combination give me? GAS.

Does not give me GUAS. Okay. Because I don't have uniform stability. This is not there. In fact, let's look at, yeah.

So this property is not there. So uniformity, no. Instability, no. But we have this guy. Okay. So the best property we have is something that I have not written here. It is something that I usually write here somewhere in between. GAS is stable plus globally attractive.

Okay. So it has a property that I have not mentioned here in this list. But you understand how it is coming.

It is not such a complicated thing. Alright. So basically it is stable plus globally attractive. Okay. So it is globally asymptotically stable. That's it. It is not globally uniformly asymptotically stable because that would require uniformity for stability which I don't have.

Alright. That's the idea. Great. Any questions? So that's what it says here I believe. Did I say it somewhere here? Yeah. Yeah. So I have actually said it here. Yeah. Globally asymptotically stable and not globally uniformly asymptotically stable. Alright. Thank you.