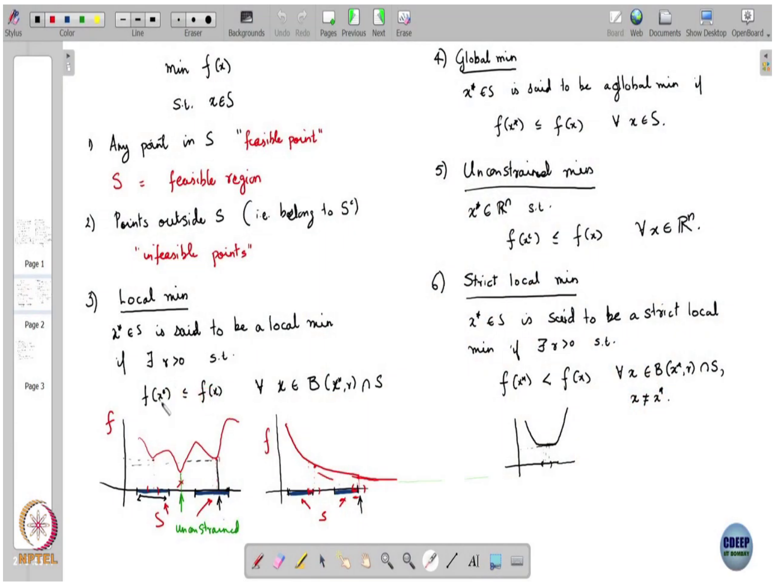
Optimization from Fundamentals
Prof. Ankur Kulkarni
Department of Systems and Control Engineering
Indian Institute of Technology, Bombay

Lecture – 3B
Different Solution Concepts

(Refer Slide Time: 00:16)



When we are looking for solutions, now that we knows is at least one solution exist, the way we have looked for solutions this is the following, we can talk off many different types of solutions; the first kind of solution is what is called a local minimum ok, a local minimum. So, x star in s is said to be a local minimum; if there exists a radius r greater than 0, such that f of x star is less than equal to f of x for all x in a ball of radius r around x star and in s, ok.

So, x star is said to be a local minimum; if there exists an r greater than 0, such that f of x star is less than equal to f x for all x in a, that lie both in s and in a ball of radius r around x star, alright. So, the way, so the what is going on here is the following; let us look at, let us plot take an example;

So, suppose my domain is like this ok; it is this interval here and maybe also and this interval here, ok. So, this is my domain my, the feasible region is this; this blue line thing here; and suppose my function looks something like this, ok. Now, what is a local minimum here? So, s is this and this and what I have plotted here is my function f.

So, you can see in this picture actually there are two local minimum; one is this point here and the other is this point here; this point is not a local minimum; why is that not a local minimum?

Student: (Refer Time: 03:30).

It is infeasible, right. So, at the very beginning, local minimum the way I would defined, it is in x star in s; so it has to at least be feasible, ok. So, this is, this point is an infeasible point. So, it is not, it is it cannot qualify to be a local minimum.

This point is feasible and moreover if I look at, if I see that look zoom in and see into a small region around it; the way the function looks is that, it has it is least value at that point, right. Likewise this point is also a local minimum, around this if I search; the function has the has its least value over there, ok.

That does not mean that the function cannot have a lower value elsewhere alright, that is a completely separate matter, ok. So, let me give you another example; just a similar sort of region, let us take s as this and this and suppose I consider a function like this; Now, this is my function, this is my set s, ok. So, now, can someone tell me where, what is my local minimum?

Student: (Refer Time: 05:23).

So, this right most point that I do you have is, this is a local minimum ok; let me draw actually the function more completely. So, function keeps decreasing further, ok. So, why is this, this right most point a local minimum? So, if I if you look at this; if you look at the function on the entire region, entire ambient space which is on the on arc; then of course the function decreases further.

But when we are talking of local minimum, we are comparing only with points that are in s, not points that are outside s; there may be points outside s that may be better than point inside s, that is not of relevance to us, ok. So, if you look at this point, this point is such that.

Now if I take a small neighborhood around this, a small ball around this and intersect that ball with s, what I am left with is only this part; only this part. And in this part it is the least, least point, this point gives you; the right end point gives you the least possible value. Any other local minimum?

Student: Right most of the left.

Any other local minimum, what is the other local minimum?

Student: (Refer Time: 06: 47).

So, if I take the right end point of the left segment, now again around this I can draw a ball like this; the ball intersects only this part and in this part this is the least. It is not at all the relevant that to the optimization problem that the function has further decreased here and so on, all of that is not relevant; because after all we are comparing only with point in s; is it clear.

The notion, there is a notion of solution that I mentioned earlier in the first class is what can be better class is more, a better name for it is what is called a global minimum. Global minimum, so is an x star in s is said to be a global minimum; if f of x star is less than equal to f of x for all x in s, it is true. So, what the question here is that.

So, you are referring I think to the right end point the of the first segment right, of the left segment, yes. So, if I take a radius r that is very large, then what was going to happen is say for example, if I take a radius large enough to include all of this suppose right, a very large segment, a larger radius; then that also you will include also this point, right. And so, consequently this will not; this will not be the minimum in such a large ball.

And this is usually the case; if you make this region larger, you will start discovering points more, you will have you will be comparing with a larger set of points. So, consequently your, it is possible that you will find a even better point than the one you started with; but a local minimum the definition is, there should be some neighborhood, however small ok, some neighborhood of positive radius in which it is; in which it is the best.

So there for me to show that this is a local minimum, all I have to do is just demonstrate for you one neighborhood; it did not work for all neighborhoods, right. So, a global minimum is defined in this way. So, now, come back to our two examples; can you point out what the global? Yeah, can you point out in these two cases; what is the global minimum? So, let us start with; let us start with this one, where is the global minimum here? So, it is unfortunately my drawing is not that good. So, let us assume this one is lower than this one. What is the global minimum?

Student: (Refer Time: 10:02).

Yeah, it is this point here, right. So, this one here this is the global minimum of the function. Is the global minimum attained in the set?

Student: Yes.

Yes, global minimum is attained in the set. In fact, this is you can see the interval, these are two closed intervals; their union that is a compact, that is a closed unbounded set and the way I have drawn the function, the function is continuous. So, the minimum the global minimum is attained in the set, ok.

Now, global does not mean what is the minimum of the function without the feasible region; means the minimum of the function which over this, over both feasible and infeasible points, that is not what global means. For that there is a different name; that is what is called an unconstrained minimum, ok.

So, I will just unconstrained minimum is simply some x star in R n; that is this is a completely separate concept. So, the unconstrained minimum need not even be a feasible point, ok. The unconstrained minimum may not even be finite ok; because you are now looking for the function, how the function, for the function even outside the feasible points.

It may even be that the function is not defined on the full regions; so all sorts of things can happen once you search outside the constraint. But assuming it is finite and finite, this is what it is referred to as ok, this is the definition,; So, let us come back to the thing that I was mentioning. So, you have a global, I was talking of global minimum. So, what is the global minimum in the first case you pointed out? It is this particular thing here.

What is the global minimum in the second? The right end point right, the right end point of the second segment here right; that is your global minimum that is your global minimum ok. The, what is the unconstrained minimum in each of these cases? In the first case in the in.

Student: (Refer Time: 12:55).

Yeah, this particular point here, this point here this is the unconstrained minimum. And what is the unconstrained minimum here?

Student: (Refer Time: 13: 14).

It is not attained, the unconstrained infimum, the value of the function is zero; if it keeps going this way, it will be zero it is, but it may not be, it is not necessarily attained, right ok, alright. So, if you see Weierstrass theorem, what it is guaranteeing you is the existence of a global minimum or it is telling you when a global minimum is attained, ok. Now, even when Weierstrass theorem is not satisfied, you may have that the local minimum can be attained and that is very much possible.

So, even though you may not, a global minimum is not attained; you may be able to find local minima that are attained ok. Now, let me go for strict local minimum. So, x star in s is said to be a strict local minimum if. So, x star in s is said to be a strict local minimum; if there exists an r greater than 0, such that f of x star is strictly less than f of x for all x in this ball of radius r around x star and lying in s, except of for x star itself.

So, if you compare with every point x that lies both in the ball and in s, except for the point case where x is equal to x star itself; then you will find that f of x star has a value strictly less than f of x, ok. In that case we say that, x star is a strict local minimum. So, again let us do and go through this example, in both of these in examples, in both these cases that we have drawn.

Let us look at the two local minima that we had, are this, let us check are they strict? So, if this one is it strict, this local minimum; this local minimum is strict, it is the way I have drawn it, it is strict, you can find it is here, you can find a ball small enough. This one also there is a curve here, you can see; if you look closely enough, this one is also strict. What about this one: this particular one?

Student: (Refer Time: 16:28).

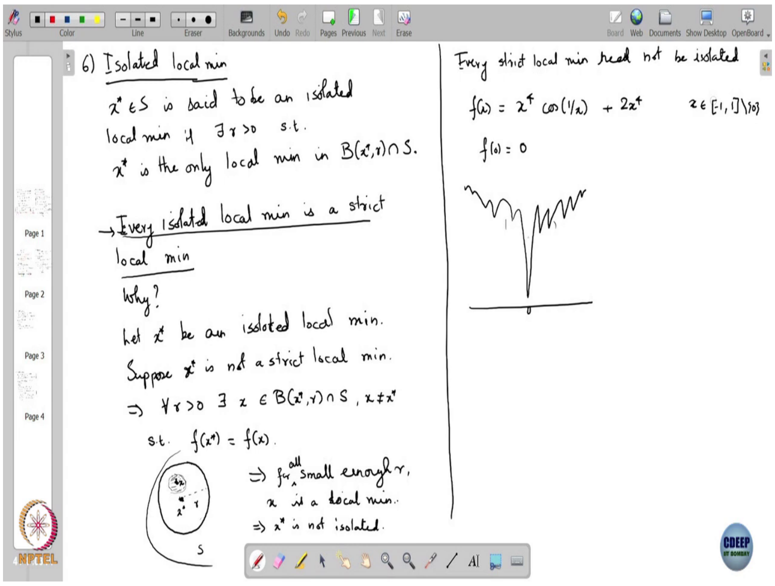
This is also strict; likewise this is also strict, ok. So, these are all strict local minima. So, what does a non strict local minimum look like?

Student: (Refer Time: 16:39).

So, for a. So, what must happen is that; so, something like this. So, all these points they have the same function value; again you take this point, you take this point, you take this point; around them you can you choose a neighborhood, there will always be another point, there is always another point that has the same; that has the same value, ok.

This is, one way in which this is one way in which you can; you have a minimum that is local, but not strict.

(Refer Slide Time: 17:40)



So, that is a very closely related notion. I will write that now, which is called an isolated minimum ok, isolated or precisely local minimum, isolated local minimum. x star in s, x star in s is said to be an isolated local minimum; if there exists an r greater than 0, such that x star is the only local minimum in B of x star comma r intersection s.

So, an isolated local minimum is one where in which you can find a small radius around that point, such that in that radius, there is no other local minimum, ok. So, now, can someone tell me, how is this related to being a strict minimum, a strict local minimum? So, clearly a local an isolated local minimum to begin with has to be a local minimum, ok.

Does it have to be strict or is strict local, the same as implies isolated or isolated implies strict? Ok. So, isolated the answer is isolated actually implies strict and the reason to see this is the following. So, when is that minimum isolated? It is isolated when in its neighborhood; there is no other local minimum, ok.

Now, suppose if it was not a strict local minimum, if you have a local isolated local minimum that is not a strict local minimum; then in that case what would happen? So, let us do this carefully. So, x star, so suppose. So, the answer is, I am going to, I will give that isolated every isolated local minimum is a strict local minimum, ok. So, this is what we are arguing. So, why is this is the case?

So, let x star be an isolated local minimum and suppose x star is not a strict local minimum. Suppose x star is not strict; if x star is not strict, then what is that mean? Go back to the definition, if it is not strict; that means that, there must be, then it is just a routine local minimum. And what does that mean?

That means that, so it was it is a local minimum means that, there exists an r greater than 0, such that f of x star is less than equal to f of x for all x in this neighborhood; Now, if it if and it is if it is strict; then there exists an r greater than 0, such that f of x star is strictly less than f of x, for all x except for x star and in this neighborhood, right.

So, if it is not strict, then it must be that, in this inequality here; equality must be attained for some point and this must be true for every radius r. No matter how small you make the radius, you should continue to get a point which is in that ball and in a that lies in that ball and in s, such that there is equality here, right.

So, what does this mean? For this means that, for all r greater than 0; there exists an x in this which is not equal to x star, such that f of x star is equal to x, ok. So, now, now let us take some radius r; now in that radius r, let us see what is going on. Here is your point x star, in and here is my. So, here is my, ok.

So, suppose; so, I am just zoomed into point x star and this is the. So, this ball does not look like a ball. So, this is my point x star, this is my radius r around it. And so, suppose here is my set s, ok. And so, for every radius r the, I can find a point x, such that f of x star is equal to f of x, a point x in this ball, right. So, here is suppose a point x, that point x.

Now, this point x has f, the value of f at this point x is equal to the value of f at x star. Now, what can I do? Now, f of x star is certainly has lower less than, value less than equal to all of these, everything in this radius, ok. So, what does this mean? This means that, so everyone is following me here; see remember x star is a local minimum, so there is certainly a small enough radius in which it is the it is the local it is the its function value is less than the function value of all other points.

Now, in every radius around x, x star also there is another point whose function value is, whose function value matches with that of f f at x star. So, which means that, if I take if I take a small enough radius; then in that radius, x star is both a local minimum ok and there is another point which matches, whose value is the same as that of at same as that of x star.

So, what does this mean? Now, around this other point now, I can draw another small ball, ok. So, what can I that and that small ball is going to be in it; that small ball is going to be inside this particular ball, right. So, all these points, their value is going to be is already less than equal, is greater than equal to f of x star, but f of x star is the same as the f of x.

So, f of x is going to be less than equal to the value of all of these small ball, all of these small ball, which would make x a local minimum, right. So, if x star is not isolated; this these new points that you keep finding, they themselves become local minimum for small enough radius. So, what this means is; this would imply that, x for small enough r;

So, for all small enough r, for all small enough r, x is a local minimum; What this means is that, x star is not isolated; So, x star cannot be isolated; So, if it is isolated, so if it is isolated and not strict; we get to a contradiction that it is not isolated to begin with, so it has to therefore be, it has to therefore be strict, ok.

So, every isolated minimum, a local minimum is also a strict local minimum; the converse is not true. So, every isolated local minimum is a strict local minimum that is what we just argued; so, every strict local minimum need not be isolated. So, here is an example of a function like this that demonstrates this. So, I encourage you to go and plot this, when you go back to your, go back home; you can plot this at in MATLAB or something.

So, consider this function f of x equal which equal x to the 4 cosine of 1 by x plus 2 x to the 4, ok. And let us take this in say minus 1 to 1 except 0; at 0 you define f of 0 as 0, ok. So, what you will find when you plot this is that this function has many undulations as you go around 0, as you go closer and closer to 0, ok.

But of course, the least possible value is at 0; but you have many many undulations as you go closer to 0, ok. So, what will happen is that, 0 is a strict local minimum; but in its neighborhood are many many other local minimum. So, in every neighborhood, there is another local minimum. So, it has, it was like this at 0; but then something like this.

So, there are many oscillations are happening at higher and higher frequency as you go here. So, this is at 0 and then in every neighborhood, I have zoomed into this; but if you go for. No matter how much you zoom in, you will keep finding another local minimum, ok. So, there will be this local minimum, this local minimum, this local minimum, etcetera.