Optimization from Fundamentals
Prof. Ankur Kulkarni
Department of Systems and Control Engineering
Indian Institute of Technology, Bombay

Lecture - 3A
Weierstrass theorem

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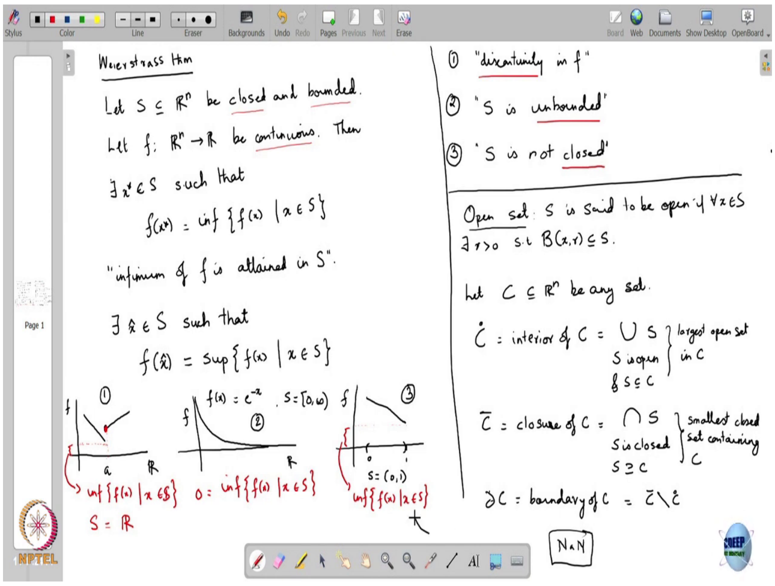

Welcome everyone. So, we will in the last lecture we ended with an important theorem which was the theorem of Weierstrass. Weierstrass was a great analyst. He had many theorems to his name. So, this is this for the particular one I am referring to is the following right.

Let S be a subset of R n be closed and bounded. Let f from R n to R be continuous. Then the claim was that there exist that a point that attains the minimum the infimum of f in the sets.

So, there exist an x star in S such that f of x star is equal to the infimum of f of x as x ranges in S.

So, in this case we said the word we used was that the infimum of f is attained in S. Now, because a problem of minimizing a function f can always be converted to a problem of maximizing another function. And when by just changing f to minus f and changing f to minus f preserves the continuity of f. So, f; if I if f is continuous then minus f is also continuous ok.

So, that does not change the continuity. So, consequently this theorem also gives us for free that there exist another x say let us call this x hat suppose, x hat in S such that f of x hat equals supremum of f over S ok. So, both the infimum as well as the supremum is attained. Now, let us just give some more intuition on what this theorem is actually doing because you will get an idea from this about what is happening when your algorithms in optimization do not converge and so on.

So, I had told you two examples last time. So, let me draw those two examples again. So, one was this an example of this function. I am plotting here on the x axis. The function is the function is a function of real numbers. So, this is what I am plotting here is just the x axis of real numbers. So, I am plotting R. So, the function is like this and there is some point a where the function value jumps to the board point there this red point that is the value of the function at a.

And then after that the function then increases this way ok. So, the function decreases till this point till a, but at a, its value is it jumps and then it increases in this sort of fashion ok. This is one case. Another case is this case which I drew last time which was the case of where the function is e to the minus x.

So, here f x equals e to the minus x and we took S as the non negative reals. So, what is the infimum of f here? The infimum of f in this in the second case the infimum of f in the second case is 0. So, as the least possible the infimum of f over S is 0. And we found that there was a problem here that there is there was no x there was no x star for which f of x star equal to infimum.

There is a now a third case on top of that. So, suppose I consider the interval from just 0 to 1, but I exclude the end points ok. So, I have an open interval from 0 to 1. So, that is my S. S is my open interval from 0 to 1 and let us just simply take this sort of linear function that keeps decreasing this way.

Now, in each of these cases can you tell me what the infimum f is and whether it is attained? So, if you recall the infimum of f in this case was this height right, this height here. This here is the infimum of f. So, I am taking S as let us say R, of course (Refer Time: 06:56). In this case the infimum of f in this case is 0. What is the infimum of f here? What is the infimum of f in the third case?

Student: 1.

The infimum of f in the third case is again this height and now in each of these cases let us ask if the infimum is attained. So, what happens in this case? In this case in the first case ok, in the first case let us call this case 1, let us call this case 2, let us call this case 3.

So, in case 1 what happens is the infimum the function is tending towards to becoming smaller and smaller as you approach a from the left, you get lower and lower values, but at a the value jumps right. So, the problem in case 1 is it you can say is a discontinuity in the in f. What is the problem in case 2?

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The problem in case 2 is that the infimum is 0, but the infimum is attained quote and quote at infinity right and if infinity is not part of our domain. We have which we are looking at only real numbers right. So, this is the problem here is that the domain is unbounded.

Domain or more specifically let us say S is unbounded. What is the problem in case 3? Is the function is the infimum attained in case 3? Again what you see here is you see that the function value decreases as we go down here. Keeps decreasing as you keep going closer and closer to 1. The infimum is that the infimum is eventually the limiting value of this height.

It would have been f of 1, if f of 1 was defined in the same way and if 1 was part of the domain, but then 1 is not part of the domain, right. So, what is the problem in this case? The problem is that the S is the boundary of S is not included in S. So, S is not closed. So, what if you go if you see these three issues that have propped up; one is discontinuity, the other is unboundedness, third is closed.

If you plug all three of them what Weierstrass theorem is telling you is that if you plug all three of these deficiencies then there is no problem. You are assured that the infimum will be attained right. So, if a function if the set is closed and bounded, if your function is continuous then you are infimum and also the supremum will of your function on S will be attained. This is what Weierstrass theorem is telling ok.

This also brings me to some couple of other points which we should mention because the jump eventually the geometry of optimization depends on the kind of sets over which you are optimizing and the kind of function over which you are optimizing. So, let me mention the couple of other things.

You remember what the definition of an open set was. An open set S is said to be open if there exist a radius r greater than 0 such that B of x comma r is the sub is completely included in S. S is said to be open, if for all x in S there exist a radius r greater than 0 such that B of x comma r lies completely in a.

Now, if a set is not open then we can in some cases extract those points around which such balls exist that is what it is called the interior of a set. So, I will define that. So, let C subset of R n be any set. The interior of C denoted in this way. It is denoted usually by C with the little circle on top.

This is simply the union of all sets S that are such that S is open and S is contained completely in C. So, take all the open sets that are completely contained in C, take their union that is what is called the interior of C. And there is an analogous notion of what is called the closure of C that is denoted by this; C with a bar on top.

This is called the closure of and this is the intersection of all sets S such that S is closed and S contains C. So, the interior of a set is this largest possible open set you can fit inside the set. Do you see that the interior itself is an open set?

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Because it is an union of arbitrary open set. So, the interior itself is an open set. So, interior is the largest. Therefore, the largest open set in C and the interior and likewise the closure is also a closed set because it is an intersection of an arbitrary number of closed sets.

So, it is the smallest closed set containing C, ok. Now, if you consider this set which is let us define this let us call let us denote this by del C like this. This is what is called the boundary of C. This is simply defined as C closure minus C interior. So, which means you have to look at the closure and remove from it all points that are there in the interior.

Whatever remains is what is called the boundary of C. Now, if C is itself closed then C closure is equal to C. If C is itself open then C interior is equal to C. Is this clear? But, if C is neither closed nor open then these may in general be different ok. Now, the reason I bring this up is because we have to as we start talking about optimization problems.

We have to pay careful attention to the kind of to the way we are defining this set. So, now, suppose you have a if you encounter an issue such as for example, the three cases that I have just mentioned. You have case 1, case 2, case 3. The way and these are in all these cases you would have the issue, you will have the difficulty that an, your infimum is not being attained.

So, there is no solution to there is no optimal solution to an optimization problem either in this case or in this case or in this case. Now, the way to fix it now in case 1, the thing the culprit is the fact that your function has a discontinuity right.

So, you have your function itself needs a change. The what you are optimizing that itself needs to be changed in order and you need to consider something that is continuous. How do you fix case 2? What will happen in case 2 is suppose you run an optimization algorithm it keeps searching for a better and better point that will give you a better and better value for it and in the process it will keep going further and further and further away towards infinity.

Eventually you will reach a stage where you have run out of your the precision of your machine. If you have used MATLAB you would have encountered this error called NaN; not a number right. So, this sort of error you this is the sort of error you would end up getting.

So, what this is suggesting is that you have your domain is too large, you need to plug your domain. Of course, this is not the only way you might want to consider a different function and so on that is a separate matter. But since you have the domain is yet to be blamed and I am talking of the domain.

So, your domain needs to be clipped and you need to clip it at some point. You need to bound it, consider a bounded domain. The third case tells you that simply bounding it without including the boundary points is not enough. You have to make sure that eventually your domain has to be both closed as well as bounded right.

So, if your boundary points are not included you again have this problem that your algorithm keeps searching, goes further and further closer to the boundary, but then the boundary is not included. So, it keeps searching, it will never actually get to a solution. So, you need; so, in the case if you are; if you have if you have in if you are in case 3, you need what you need to do is make sure that.

So, if you are in case 3, one simple way of fixing this would be that you simply you make sure your set is closed and for that all you need to do is simply take the closure of your set right, ok. So, now, let us again go back to an optimization problem and now from here onwards I am going to assume that an infimum is attained.

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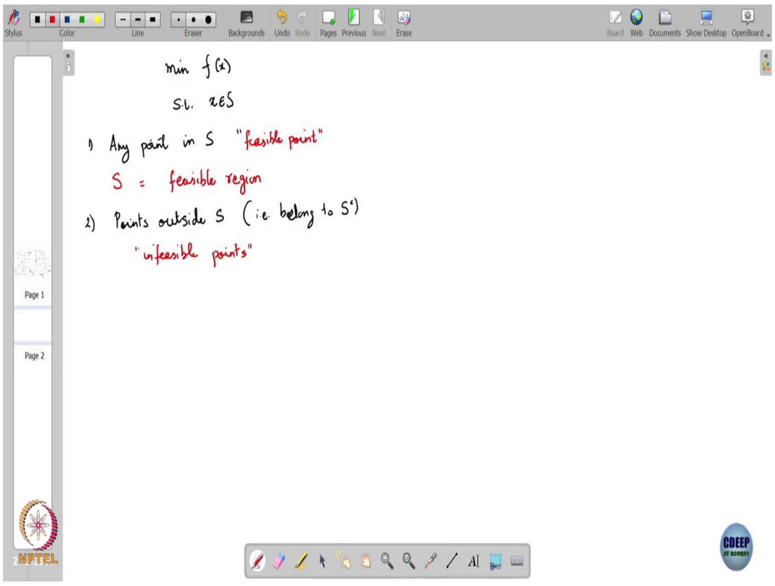

So, if you remember I my notation for when the infimum is attained we said we will write it like this, we will call it the minimum. And so, the way we will write the optimization problems is that we will write here a function f that we want to minimize, subject to this condition. So, when we when I write subject to something it means that this is the set over which I am minimizing and this is the function that I am looking to minimize.

So, in this any point in S is called a feasible point, it is called a feasible point. The S as a whole is called the feasible region. Points that are outside S that means that is they belong to S compliment they are called infeasible. They are infeasible points.