Optimization from Fundamentals
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Lecture - 24C
Optimal control for a system with linear state dynamics and quadratic cost

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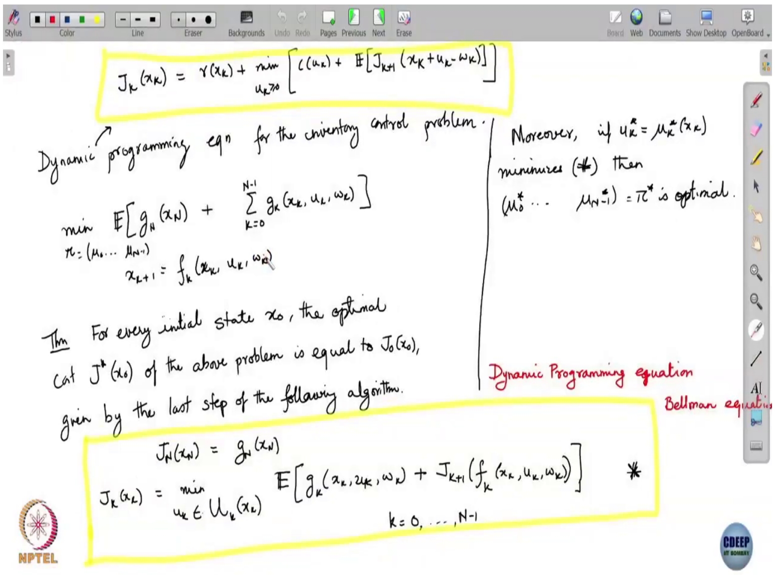

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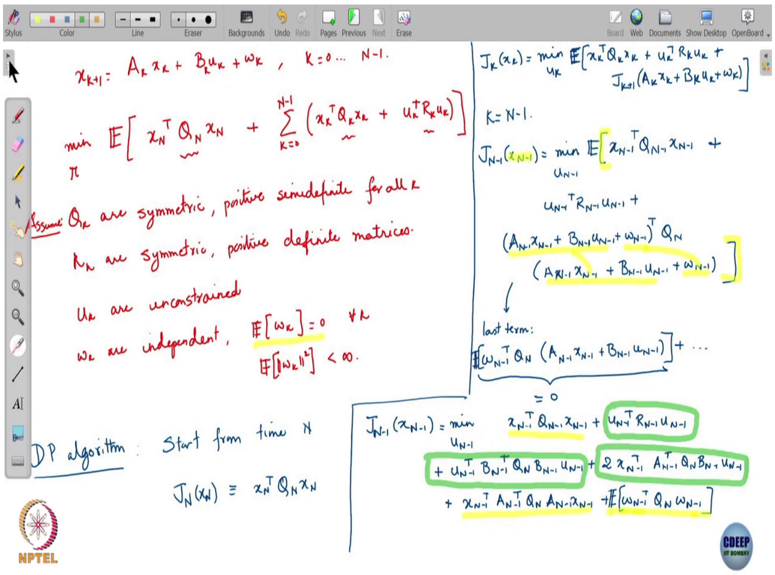

Ok. Now, let us now take the special specific special case. The special case is this. So, now, my dynamics are given in this sort of form. I have x k plus 1 is given as a function x A k x k plus B k u k plus w k ok and this is we define this for K equal to 0 till N minus 1. The cost function which we want to minimize, we want to minimize this cost function overall such overall policies. Cost function in is takes this form.

So, the terminal cost is quadratic in the terminal state in this form. So, its x N transpose Q N x N and the cost at each stage the stage wise cost are also quadratic and in this sort of form. Its x k transpose Q k x k plus u k transpose R k u k. Now, the matrices here are of appropriate dimensions.

So, we should we assume that these matrices Q N, Q k, R k these are all square matrices. A k, B k are matrices of appropriate dimension. So, that they so, that this state equation can actually the dynamic state dynamics can actually be written out properly. Now, we assume that these Q k's are.

So, assume that this Q k are symmetric they are symmetric and positive semi definite matrices ok and for all k. And also assume that R k are symmetric and positive definite matrices. So, Q k's are positive semi definite, R k's are positive definite right. Now, the there are no constraints on the control actions.

The u k's are unconstrained. So, we have we do not impose any constraints on the control action. The constraints actually would make the problem somewhat complicated. We are not mentioning any putting in any such constraints that the disturbances or noise w k are independent ok.

We are not we are also not assuming that they have any particular distribution such as Gaussian or anything like that they are just independent ok. But we will assume that their means are 0 ok. So, we will assume that they have been centered that means the mean is 0 and we will also assume that their variance is finite ok. So, the second moment is finite alright ok.

So, now so, this is a this is a very popular formulation of the problem. See these problems arise all the time in control theory in inventory management and so, on. The quadratic cost is often is a very reasonable is a very reasonable cost because it many cost can be modeled in this sort of way where one is trying to say orient minimize the distance from a particular target or minimize the deviation from a certain from a certain nominal value etcetera.

A lot of these errors end up taking the form of the errors that we want to minimize often end up taking the a quadratic square quadratic type of form. So, that is the that is one of the motivations for this particular problem ok. So, there are many generalizations of this particular problem also I am not looking at any of those generalizations. Let us look at this simplest one much more general problems can be considered.

For example you can also have terms that are that involve a mixture of these Q u Q x k and u k in the in your cost. So, we are ignoring such terms here and for simplicity it makes it does not it does not add too much more to the problem alright ok. So, now, let us apply the dynamic programming and let us now apply the dynamic programming algorithm yeah ok.

So, what is the; what is the approach? We would have to start from the last time instant ok and over there we define simply the value function to be simply the terminal cost. So, we define J N of x N to be identically equal to x N transpose Q N x N alright. This is our this is the initialization for the dynamic programming for the dynamic programming algorithm ok.

Now, what we need to do is do this what would be the what would be the dynamic programming equation at each time step k? So, let us write J k of x k that is going to be equal to the minimum of u k of the expectation of x k transpose Q k x k plus u k transpose R k u k plus now J k plus 1 A k x k.

So, let me write this a little neatly. Plus J k plus 1 of; now remember I am I need to I am going o substitute in J k plus 1 x k plus 1. x k plus 1 is given through this equation. So, I am going to have substitute this as A k x k plus B k u k plus w k alright ok. So, now, let us do this. This does not help us much because we do not know what the form of J k plus 1 is.

So, but let us put do this more concretely. So, let us so, what I want you to observe is that through these calculations they may seem a little you know kind of routine or mundane first, but the thing I want you to observe is the following. That it will turn out that when I since that J k plus 1 would be quadratic and that would ensure.

The, if J k plus 1 was quadratic was a quadratic function then this entire function that we have here the entire function that we have here would end up becoming a quadratic function of u. And once it is a quadratic function of u minimizing it in and moreover it would be A positive it would be a convex quadratic function because of the choice of because R k's are symmetric and so on and so forth all of that would ensure that this is actually a convex quadratic function of u alright.

And when I when you minimize it will turn out that you would get u k to be a function. You would get u k by setting simply the derivative of that quadratic to be equal to 0. And on solving that equation where the derivative is set equal to 0 the u k the optimal u k u k star would be a linear function a linear function of x k ok.

So, when this J k is quadratic you can imagine that the only terms the only cross term between x k and u k would be coming from this and those also would basically involve u k a term where the degree of u k is 1 and the degree of x k is 1 putting that together you would get u k star as a function of as a function of x k and it would be a linear function therefore, alright.

So, let us illustrate this for K equal to N minus 1 ok. So, let K be N minus 1. So, at K equal to N minus 1, I will now I am now referring to J N minus 1 of x N minus 1 as equal to the minimum over u N minus 1 of this x N minus 1 transpose Q N minus 1 x N minus 1 plus u N minus 1 transpose R N minus 1.

So, this is I just write this a little more compactly very nicely here plus u N minus 1 transpose R N minus 1 u N minus 1 plus. Now remember J N ok J N is here J N is a quadratic function already ok. So, for K equal to N minus 1 I have got I what I got what I wanted which is simply that I which was you know I wanted J k plus 1 to be a quadratic function.

So, for J for K equal to N minus 1 J N which is J is already a quadratic function. So, all I need to do is just I put this in. So, now, I am going to put in x N. I am going to substitute for x N from here using this ok. So, I am going to get A N minus 1 x N minus 1 plus B N minus 1 u N minus 1 plus w N minus 1 whole thing transpose Q N the same thing A N minus 1 x N minus 1 plus B N minus 1 u N minus 1 plus w N minus 1 bracket close ok.

So, the expectation is now of this entire term. So, remember the expectation the bracket opens here and it closes here. So, there are it is a sum of all of these terms. Out of all of these remember x N minus 1 is just the term x N minus 1 is simply a nominal initial state.

So, it is not a random variable, it is not a true state realized during the problem, it is just a nominal state that any candidate state that we are starting off with alright. So, and so, now and we are now finding u N minus 1 as a function of that x N minus 1. Now, as you can see here we this the; so, let us try to simplify a few things here you can.

So, the last term here that we have this last term this term can be expanded ok, this can be expanded. So, the if you expand this term out. So, let me write out this the last term ok. The last term when we expand this out let us see what kind of form it takes. See it would take the following form. I would get a term which has w N minus 1 transpose Q N times this whole thing A N minus 1 x N minus 1 plus B N minus 1 u N minus 1 right.

And now the last term has this plus several other terms. Now, what can we say about this particular term? So, notice that because I as I said x N minus 1 is simply some constant or some deterministic nominal value that we have taken and u N minus 1 is also deterministic, it is a it is simply a function of x N minus 1 right.

So, as a result of this the only randomness here is in w N minus 1. Now, when I take the expectation of this particular term. So, let me write out the expectation of this. What would be the expectation of this? This expectation would necessarily be 0 right. So, this term would have expectation 0 and that is because w N minus 1 is remember we assumed that all the w's are of all the w's are of mean 0.

So, this term here would have expectation 0 ok. So, this term would be 0. So, what you would be left with? So, essentially the only kind of terms you would be left with are the following. You would have a term that in that is this term multiplying this term via Q.

You would also have a term in which w N minus 1 multiply w N minus 1 via Q right. So, there will be terms that are quadratic in yeah quadratic in these two in this and quadratic in this. So, this is the sort of term we would get alright. So, now, let me let us actually write that out.

So, J; so, as a result we have J N minus 1 as a function of x N minus 1 that is equal to now minimizing minimization of over u N minus 1 of the expectation of x N minus 1 transpose Q N minus 1 x N minus 1 plus u N minus 1 transpose R N minus 1 u N minus 1 plus u N minus

1 transpose B N minus 1 transpose Q N B N minus 1 u N minus 1 plus two times x N minus 1 transpose A N minus 1 transpose Q N B N minus 1 u N minus 1 plus x N minus 1 transpose A N minus 1 transpose Q N A N minus 1 x N minus 1 plus the final quadratic term which would be w N minus 1 transpose Q N w N minus 1 ok.

So, let me run you through what these terms are. These terms are simply. So, the last term here is the product of w N minus 1 transpose Q N w N minus 1. So, that is this term here. The other three terms are this transpose Q N times itself times this term again and that again has been expanded out. What you get is as a result you get a quadratic term in u N minus 1, you get a quadratic term in x N minus 1 and you get this cross term.

So, remember this cross term. This is with u N linear in u N linear in x N that is the sort of term you get. So, what is the lesson here? The lesson here is that now what whatever is there in the bracket let us look at what is all in the bracket and there is stuff which is deterministic and this stuff which is random here.

The only thing that is random here is actually just this w N minus 1, this is the only quantity that is random right. That is because x N minus 1 is fixed u N minus 1 is obtained is deterministic as a function of x N minus 1. So, the only randomness is here. So, I can actually move my what I can do is I can move this expectation from here all the way till just here.

So this is great because now I have this is the only expectation ok and moreover this term that I have underlined this is also has nothing to do with u N minus 1, it is a constant. It is a it is a it is an additive constant that you are picked up right. Now, what else is a constant here? Let us look at this. So, you can see this term here as is constant as far as minimization is concerned ok.

So, this was the only random term. So, it was all the other terms of constant for as far as the expectation was concerned. This is a constant as far as the minimization is concerned. So, it has been moved out. This term is also constant as far as minimization is concerned because it depends only on x N minus 1.

This term is also a constant. It depends only on x N minus 1. This does not affect the minimization right. So, I have therefore, left; I am therefore, left with only these three these two or you can say these three other terms ok. There is this one which is a quadratic in. I am left with just these three other terms which are going to be which are my which actually affect my minimization. The terms are this one this one and finally, this one.

Now, let us observe these green boxes. You can see, these are there is something to be noted here these are actually quadratic terms. So, the u this term here u appears in a quadratic form. Here also u appears in a quadratic form. This is a bilinear form when which this is linear in u as well as in x N, but point is as far as the minimization is concerned it is just linear in u.

So, this is therefore, a quadratic function of u and what kind of quadratic function? Well, you look at this the Hessian here the Hessian here is R N minus 1 these are supposed to be symmetric positive definite matrices. So, this is a positive definite matrix. The other Hessian here is B N minus 1 transpose Q N B N minus 1. You can verify easily that this is all this is now a positive this is a positive semi definite matrix.

So, this is a positive semi definite matrix, this is a positive definite matrix. So, consequently what we are minimizing is actually a strictly convex function strictly convex quadratic function of u N minus 1 ok. So, its solution can simply be obtained by setting the derivative equal to 0.

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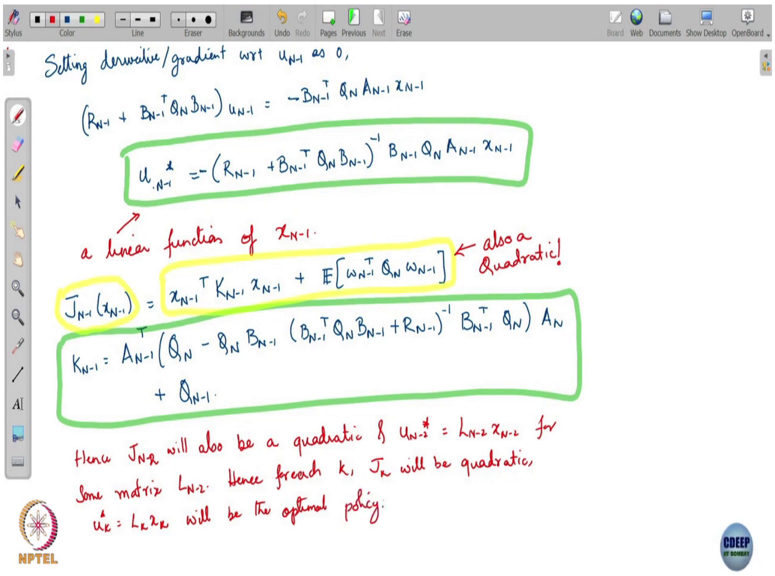

So, setting derivative or gradient with respect to u N minus 1 as 0, what we get is actually this equation. We get R N minus 1 plus B N minus 1 transpose Q N B N minus 1 whole into u N minus 1 equals negative of B N minus 1 transpose Q N A N minus 1 x N minus 1 ok or in other words u N minus 1 star is equal to this.

R N minus 1 plus B N minus 1 transpose Q N B N minus 1 the whole inverse; let us put a negative outside here B N minus 1 Q N A N minus 1 x N minus 1. You can see what we have got here we have got that u N's u N minus 1 star is a linear function ok. So, this is a linear function of x N minus 1. So, what this means is that your the optimal policy at a times step N minus 1 should take the form some matrix multiplied by the state at that time alright.

Now, and you substitute this back in you actually can evaluate also you can also evaluate J N minus 1 which is the value function as a function of x N minus 1 that turns out to be taking that turns out to take this form it is sum x N minus 1 K N sum K N minus 1 times x N minus 1 plus this expectation of w N minus 1 transpose Q N w N minus 1.

Now, what is this K N minus 1? This K N minus 1 is actually something that we ok I will explain what is K N minus 1. Or where did I get that last trailing term? The last trailing term is actually nothing but this term. This was the; this was the trailing term that you picked up. The K N minus 1 is what we will get constructed using this quadratic in x N minus 1, this quadratic in x N minus 1 and the optimal value that you would get after substituting u N as u N star that will become that being a function of x N minus 1.

So, since u N star is a linear function of x N minus 1 all these green terms will combine together to give you another quadratic in x N minus 1 right. So, this will also become a quadratic in x N minus 1. You can check that this the expression is like this that K N minus 1 is equal to A N minus 1 into A N minus 1 transpose Q N minus Q N times B N minus 1 times B N minus 1 transpose Q N B N minus 1 plus R N minus 1 the whole thing inverse B N minus 1 transpose Q N whole thing into A N plus Q N minus 1.

This is K N minus 1 right. But what is the lesson here? The lesson now is that your the value function now at time step N has also turned out to be quadratic ok. So, the value function at time step N has also turned out to be a quadratic function and so, this. So, what, also a quadratic. So, what we got?

In the previous time step what did we conclude? When if the value function at time N was quadratic then the value then the value function at time N minus 1 also ended up becoming when the value function at time then in computing the value function at time N minus 1 all you had had to do was set derivative equal to 0 and you got a linear policy.

Moreover, what is happening now is that the value function at time N minus 1 is also turning out to be a quadratic. This is also turning out to be a quadratic. So, as a result now the value function, if I now put this back in into step N minus 2 then I would be able to repeat similar calculations, but if I put this back into step N minus 2 then also my value function at time N minus 2 would become quadratic and the policy at time N minus 2 would be linear in the state at time N minus which is x N minus 2.

So, the; so, u N minus 2 star would also become a linear it should be would be linearly dependent on x N minus 2 and this would go on. And I would therefore, be able to recursively work back all the way till time step 0, where again I would be solving some sort of a some quadratic optimization.

It is of course, true that your the Hessian of these quadratics and a coefficients involved all of them continue to get complicated, but that is much better than having a non quadratic optimization right. So, the beauty of this particular problem structure is that this there is this inherent invariance that you start off when you have a quadratic cost and linear dynamics the there are lot of nice coincidences that come into play in such a way that the complexity that of the problem does not change as you go backwards through the dynamic programming algorithm.

The you continue to keep getting quadratic function quadratic optimizations and you continue to get linear policies right. So, this the this helps you very elegantly find the optimal cost function optimal value and optimal policy right. So, as so, this can. So, to for completeness let me state this here.

So, hence J N minus 2 will also be a quadratic and u N minus 2 star be linear right. It is like this, L N minus 2 times x N minus 2 for some matrix L N minus 2. So, for I will give you the formula also for L N minus 2. So, this there would be therefore, this kind of a relation right. And moreover hence this will also be quadratic, hence for each k J k will be quadratic and u k equal to L k times x k will be the optimal policy.

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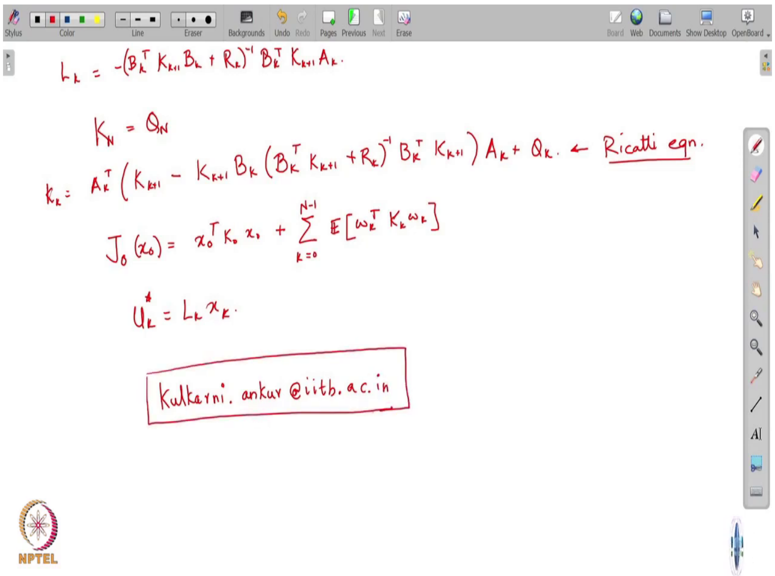

So, what is this L k? The L k is this term the L k actually can be given this way L k is negative of B k transpose K k plus 1 B k plus R k. This is a capital K this is a small k please note inverse B k transpose K k plus 1 A k. What are these K k's? They are K at time N is just simply Q N and the K's at every time can be computed recursively in this way.

So, K at times small k is equal to A k transpose K at time k plus 1 minus K at time k plus 1 B k into B k transpose K at time k plus 1 plus R k whole inverse B K transpose K at time k plus 1 the whole again into A k plus Q k. And what would be the optimal cost? The optimal cost is J 0 of x 0 which is equal to x 0 transpose K 0 x 0 plus. So, this is your the initial, the cost due to your initial state and plus these terms that you were picking up at each step.

You recall we were picking up one quadratic variance like term for the w's. So, that is exactly the term that you would accumulate at each step. This would be the optimal cost as the function of x 0. So, it turns out that the optimal cost is also quadratic as a function of the initial state right. So, very several nice things happen as in this problem. The optimal policy turns out to be linear. So, this is so, u K is equal to. So, maybe I will write it here.

So, the optimal policy is the optimal policy is to simply apply L k on x k just this lin application of the linear of a matrix on the state right and state the matrix itself can be computed through this matrix K k. The this particular equation is what is called the Riccati equation ok. This equation is what is called the Riccati equation. The through that equation you can recursively compute the K k's and substitute them back and to get your optimal policy.

Now, here there are a few other things to note here which are not related much to optimization, but something more again about this the structure. See notice that L k has no dependence on the w's. You notice the way the noise appears in the problem is that it somehow drops out of a lot of the calculations and only keeps adding up as a as you know an additional term here.

So, if the noise was deterministic means, the variance was 0 this term would vanish, this last term would vanish and you would be left with just. You will you know you what you would be left with is just simply, this particular term; x 0 transpose K 0 x 0 where K 0 would have to be calculated recursively through this Ricatti equation.

But look at the look at the amazing result here that u K is u K star the optimal policy to take that a optimal policy or the optimal action to be applied is simply L k times x K. So that means, whatever be the state you just apply this particular matrix and this matrix itself has no dependence on the noise on the variance of the noise.

So, the noise makes no appearance in the calculation of this matrix. So, if you see the K k recursion the only things that appear here are the R's, Q's, the A's and the B's and the w's have make no appearance here whatsoever. So, which means that the Riccati equation is the same regardless of whether you have noise in the system or you do not have noise in the system.

Which also means therefore, that your feedback or the optimal policy that you have to choose is also the same regardless of whether you had noise in the system or not. This is again an amazing miracle actually that comes about due to this particulars problem structure.

It is effect its effectively telling you that you can pretend as if there was no noise whatsoever and continue to apply the same control that you would have or the continue to apply the same policy that you would have in the absence of noise ok, in the absence. The policy that you would have applied in the absence of noise you just continue to apply the same policy in the presence of noise, but on the state that would get realized in the new system.

Of course, you do not take the same actions necessarily because the state itself would be different, but the plan or the strategy or the policy that does not change due to the presence of the noise. So, the only way the noise affects is that it keeps adding this little offset term to your cost function to your optimal cost which depends on the variance of variance of the noise.

So, it keeps adding this little bit of a you know an error due to a error due to noise in some sense or an offset due to noise, but it does not affect how you are going to steer your system alright. So, this, because of all of these reasons this particular problem structure is a pet problem and favorite problem across many disciplines operation research control and so on.

Many people have applied this. In fact, much more deeper coincidences are come in to play in more general problems again with quadratic cost and so on and that is a that is a subject for later. So, with this I am also I also end the content of this course.

There is this these forms essentially gives you a way by which you have you can relate deterministic optimization and dynamic optimization. There are other ways also of relate relating the relating deterministic and dynamic optimization those are for instance there is something called as a minimum principle of dynamic optimize dynamic control problems or dynamic optimization problems and that can be obtained through KKT conditions in KKT conditions.

That something that is beyond the purview of this course, but this is something that you can look up ok. As my all my credentials are there on the NPTEL website. If you would like to get in touch with me about any problem you are working on if or if you like to talk about anything feel free to get in touch with me you can look me up also on the internet.

So, this brings us to the end of our course on optimization. We have covered vast and vast domain. We have started from unconstrained optimization and gone on to constraint optimization, KKT conditions and finally, we have ended up with dynamic optimization. But, the subject itself is much vast is much larger than all this. It is an extremely evolved and well developed subject. If there is anything you would like to talk to me more about feel free to get in touch with me.

You can look me up on Google. You can simply search for my name on Ankur Kulkarni on Google and my webpage should show up there you will find all ways of contacting me. You can also you can email me at on Kulkarni dot ankur at iitb dot ac dot in. So, if there is anything you would like to talk, I will be happy to respond. Do get in touch.

Thank you for this course.