Optimization from Fundamentals
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Lecture - 23A
Dynamic programming: Inventory control problem

Welcome everyone. So, today I will start with a new topic which is also the last topic of our course on Optimization. The topic is called Dynamic programming or dynamic optimization. The theme of dynamic optimization is about taking decisions over time at multiple time instants one after the other and you optimize not just the cost at each time, but a consolidated cost over the entire time horizon, that one is considered.

So, in as compared to optimization in which there is one cost function that is to be optimized at that particular time. The in dynamic programming we have a cost function at each time and what we want to optimize is the sum total of these cost functions over the entire time horizon. The complications in dynamic programming arise because this is not simply a separate optimization at each time step.

Because the decisions you take at one time step impact the information that you have at the next time step and the decisions that you will take at the following time steps right. And as a result of this what one has is actually a series of optimizations that are coupled intricately with each other where there is where decisions of the past feed into the decisions of the future.

As a result of this we have this kind of optimization requires a study in its own right; it is not merely a corollary of static optimization. And as a result we you know it requires in a kind some viewpoints and tools that that were not that are not actually present in static optimization ok.

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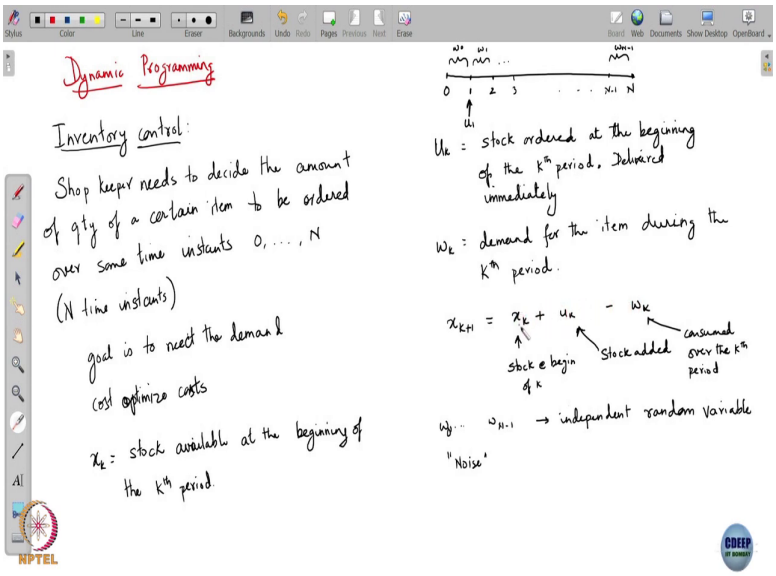

So, to motivate and to give an to explain what I mean by all this is let us take an example ok. The example; the topic for today is what is called dynamic optimization so or dynamic programming alright. And what is my example? Example is what is called inventory control ok. So, the example is of what is called inventory control.

Now, inventory control rarely refers to the situation where you are say a shop shopkeeper or a shop owner and he wants to decide the amount of quantity; needs to decide the amount of quantity of a certain item to be ordered over some time instants; some time instants.

Then we will say our sometime instants say 0 to N ok. So, these are N time instants; N time instants and you want to order quantity O at each of these time instants alright, then you want to the; and you want to determine what is the quantity to be ordered. So, now, what is the, what are the considerations involved in a problem like this? Well you want to order a certain amount of quantity the goal is to meet the demand ok.

So, the goal is to meet the demand; is to meet the demand it is also to optimize costs. You may also have other constraints such as for example, you may have storage constraints you may not be able to store an you know very large amount of quantity of that particular item, you may there may also be constraints of say perishability that the item may be may need to be disposed of before a certain time and so on ok.

So, what we will let us just to formulate this problem let x k ok denote the stock available at the beginning of the kth period ok. So, now are we so what we will do is we will be making decisions over time periods ok. So, and those, but then we the time p are time from time starting from 0 to N is actually to be divided into N periods. So, this is period 1 this is the end of period 2 etcetera etcetera, alright.

This is how we will think of time so time for us is going to be slotted and discretized like this. So, we have to develop a convention about when exactly are we keeping track of the state or the stock available to us ok the stock. So, our convention is going to be that we will keep track of the stock available at the beginning of the kth period right.

So, the when we are talking of stock x k it is going to be or say stock x 3 it is at the it is going to be at the beginning of period 3, right. So, that is the so this is the convention that we will have to we will adopt. So, you could also adopt a different convention in which you take the stock at the end of the period, but we our convention is going to be that its going to be at the beginning of the period alright, ok.

So, the stock and let us denote u k as the stock ordered at the beginning of the kth period. Now, once again this is again a convention are we will; we are keeping the convention that the stock is being ordered at the beginning of the period so it is at the left hand point of the period alright. And now then depending on the problem you may have complications such as the stock may take some time to get delivered once you order it say for instance you order it and it comes to you in after a certain number of period.

But, for simplicity I am going to assume that the stock is both ordered and its in the beginning of the period and delivered also immediately; delivered immediately. So, this would mean that our time periods are wide enough so that they take into account both the amount of time that we are measuring, but also the delivery time is insignificant and then right.

Let, w k be the demand for the item during the kth period ok. So, it is the demand of the item during the kth period. Now, demand again is not something that is not something that we can attribute to either the left end point or the right end point its rather a property of the entire period, but the way since we are adopted this convention what we can say is that we can relate the stock available at the beginning of the k plus 1th period to the stock available at the beginning of the kth period and the demand during the kth period right.

So, now as a result of this we have that x k plus 1 is going to be equal to x k plus; so this is the stock available at beginning, this is the stock you have added or ordered, this is the stock consumed over the kth period. So, by the time you reach the left end point of the K plus 1th period ok; by the time you reach the left end point of the K plus 1th period you reach the inventory level would then be x k plus 1 ok or the level of the stock is x k plus 1 alright.

Now, we are going to assume that w's w 1 the these w's let us say w 0 to w N minus 1 right and these w 0 to w N minus 1 these are let us say independent random variables. But, if you are not aware comfortable with random variables its ok you can simply think of these as some exogenous variables that we do not have we do not know the value of. So, if you now the important thing here is that w is realize the w k is going to be realized during the period k right.

So, w 0 gets realized in this period this is w 0 its realized here w 1 gets realized here etcetera etcetera, right. And w N minus 1 gets realized here at the at the beginning during the N th period ok which starts at its during the N th period or during this last period here ok. So, these are going so now, because this is going to be realized during this period when we are making this decision ok; when we are making the decision of ordering say a quantity u 1 at time 1 ok.

When we are making this decision of ordering quantity u 1 at time 1 or a quantity u 0 at time 0 or whatever the that we are not aware at that time of this of the value of w that will be realized over this particular period. So, u 1 has to be decided without the knowledge of w 1 right of course, you would be aware of u 0 but not their value of u 1 right.

So, this has to be done without the knowledge of; without the knowledge of this particular the realized demand at that time. So, this is this particular thing because this is something that we so its exogenous and is its distribution we cannot control its what we use the word noise for this; we call this noise. Noise is simply any randomness which comes from the environment whose distribution is cannot be chosen based on actions that you that we can take right.

So, again as if you are not comfortable with probability randomness noise etcetera it is alright, but just be aware that in taking in solving these problems we have to bear in mind that x k is not; x k is to be chosen before the value of w k is known alright, that is something that you have to bear in mind alright. So, let me go to the next page then. So, we these decisions have to be taken in order to optimize a certain cost. So, let me write out let me write out a cost function for.

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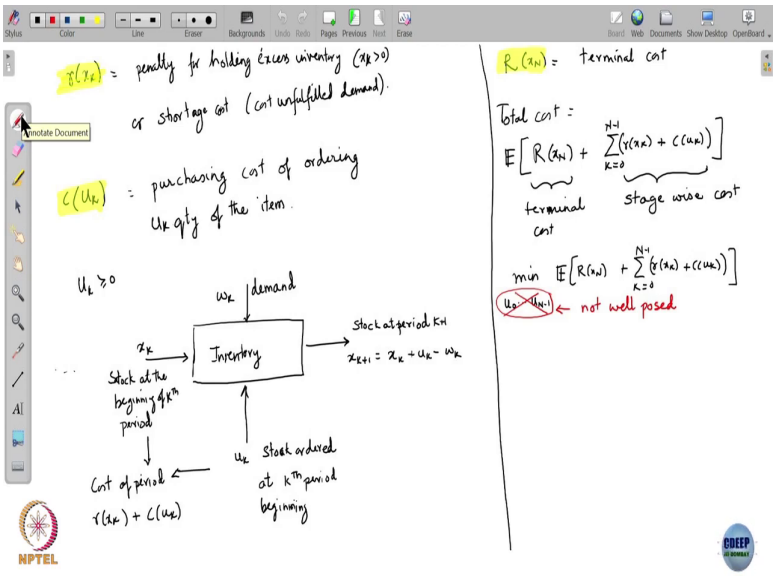

So, let us say there is a cost r ok r of x k and let us; what is this? This represents a penalty for holding for either holding excess inventory; inventory in the that is in the case when x k is positive when you have; when you have a positive amount of stock left with you or a shortage cost. So, this is the cost say of unfulfilled demand. Let us say a some notional cost we have for unfulfilled demand ok.

So, how does this so in terms of a block diagram if you want to think about what is going on here sorry there is another cost term which is say a purchasing cost ok. So, when every time we order u; quantity u k let us say we incur a cost say c times u k or c of u k. Purchasing cost of ordering u k quantity of the item.

Now, these are all kept simple and of course, my r could also depend on k here say this r here could depend on k the purchasing cost could also vary with time it could vary with its here its

just varying with the quantity, but it could also vary with time right. A natural constraint since we are talking of purchasing an item not of disposing of items.

So, the natural constraint then is that u k should be greater than equal to 0. Now the so what is the, if you think of what is going on here in terms of a block diagram? So, you have your inventory. Now, it gets in demand w k, the stock at the beginning of kth period that is x k. Then there is also the u k which is stock ordered at kth period; kth period beginning, once again this is also at the beginning right.

Then you have the cost of period k that corresponds to period k so that for us is going to be r; r of x k plus c of u k alright, and now with this information you then putting in this information the system then moves to the next time period and you get to stock at period k plus 1 in the following way x k plus 1 is x k plus u k minus w k.

So, remember once again that u k has to be u k here has to be chosen without knowing this w k although these arrows are all pointing in at the inventory system here remember u k has to be chosen based on x k and the past w k is not as a function of this; this w k ok.

So, now we the goal then is, now there are what is the objective? Then the objective is to say well I want to find; I want to decide how much stock should I be ordering in order to minimize my total cost. So, let us say there is also in addition to these cost let us also assume there is also a terminal cost r of x N this is let us call this a terminal cost; a terminal cost is simply a cost that you would incur at the end of the time horizon that you are considering.

So, when you reach this point N here at the of the time horizon that you are considering you have no more decisions to make that is the; that is the time up to which you are looking ahead and taking decisions once you reach that sort of time that end, if you are left with an inventory of x N, what is the; what is the cost that having that inventory would carry?

So, either positive inventory or negative inventory what is the cost that that would carry? That is what we called the terminal cost. So, our total cost then is the sum is comes up as a sum total of 3 terms. So, you have your cost of that you have for holding you have your cost of

purchase and this cost apply these two cost terms apply at every time instant. But, then in addition there is also a terminal cost here at the N th time instant right.

So, the total cost then, total cost that; total cost that we incur is equal to E x expectation of the terminal cost so plus the cost that you incur at each time period. So, there are these N time periods K denote K equal to 0 to N the cost you incur at each of those time periods is r of x k plus c of u k right.

So, this here is the stage wise cost a cost for yet each time instant this is terminal cost. Now, what we want to do is decide what should be these actions ok or what is with that what we want to decide is, what is the amount of stock we want to order at each time instant at each of the k at each of the k at the beginning of each of the N periods denoted 0 to N minus 1 right.

So, the amount of stock to be ordered is our decision. So, you would what you would want to do is minimize the above cost minimize this R of x N plus this sum. We want to minimize this, but then minimize by doing choosing what you would like to minimize this by choosing the amount of stock to be ordered at each time. Now, but if I there is a little bit of a catch here and so that is one thing I want to make sure you all of you understand.

So, if I write here say my action that I want to choose which is u 0 to u N minus 1 well that is correct? Yes, I do want to choose these; choose the quantity that I want to order at each time instant. The trouble with defining it trouble with writing this here is that these quantities are to be decided based on the inventory level that we would see at the beginning of the time period.

So, u k is the stock to be ordered at the beginning of the kth period and that would; obviously, be decided once you look at the inventory at the K th; at the kth period and that inventory at the kth period would depend on what demand has transpired between up to that time from the start of; from the start of the first period right.

So, the u's u k is not simply a quantity that I can decide at this right now because it depends on these inventory that would be available. And the inventory that would be available

depends on the demand that would get realized all these demands from the previous time instants here.

So, there would be previous time instants here there the demand from those time instants would determine the inventory that would be available at the kth time instant and the demand that would come in the previous time periods is not something that I know at the start of the; at the start of the problem right.

So, when I am trying to make this all these; when I am trying to decide this is this has been this problem has been pose assuming I am sitting at time period 0 and looking ahead into the N time periods that I have; that I have considered. And since I am looking ahead, I do not have knowledge of what the exact demand is going to be, I do know its probability distribution I know what kind of what its likely to be with what probability.

But I do not know what its value is going to be right. So, as a result because I the information on which my action is to be taken that is information on based on which my stock is to be the stock to be ordered is to be decided since that information itself is not available to me at the start of when this decision when this problem is posed I am actually in a fix, because I do not know what action should I be planning what kind of information I should be planning for right.

So, the value of the amount of stock to be ordered has to be decided; has to be decided only once we have seen the information that is available, which is the stock that is available at the beginning of the period. So, without that being provided to us the amount of stock to be ordered is not something we can decide right.

So, as a consequence this way of writing the of posing the problem is actually not well posed. So, this is incorrect this is not well posed; not well posed. And the reason this is not well posed is because I do not have a way of even talking of the action you; the action or the stock to be ordered at time k sitting at time 0 because the information on the basis of which that action is to be taken is not available to me at time 0 ok.

So, what is this mean then? So, this means that we are start we are thinking of this problem sitting here at time at the start of the time horizon. We know there are going to be demands that are going to be realized over these period but they their exact that their value is not known to us we just know their distributions probability distributions.

Sitting here we want to still be able to decide what is it that we need to what action should we be taking or what amount of stock is to be ordered at each of the time period right. So, because and a specific action cannot be decided because we do not know because that action would depend on the inventory available and that inventory available in turn would depend on the demand that would get realized which is not which in turn is not known to us right.

So, as a result of this what do we; what do we see we see that when we have to make these decisions to minimize the cost that evolve that is stated over the entire time horizon ok; a cost that is stated over the entire time horizon this cost we cannot possibly; we cannot possibly pose this problem as a problem of deciding the amount of stock to be ordered.

The plane and simple reason is that the amount of stock to be ordered depends on the information that will come up later during the problem itself not at this and is not available to us at the start of the problem. So, what is then the alternative, what is the alternative? Since we are we can; since the quantity to be ordered is going to be decided based on the information that is a that will become available during the problem, what is the alternative, what is it that we can actually decide at the start of the problem at the beginning of even before the; even before the demand is actually realized?

The thing that we can decide is not the quantity to be ordered, but what our plan is going to be for quantities that we would want to order based on the information that we would potentially get during the course of the during the course of the problem or during the course of the time during course of time during the problem right.

So, we can decide what is the how much a quantity would be ordered if we had a certain amount of stock available at the start of the start of the time period. That means, what we can

do is not decide a specific quantity, but rather an entire plan which tells us how much quantity should be ordered for each level of stock right. So, if this level of stock is 100 units you would order so much quantity, if the level of stock is 20 units you would order so much quantity 30 units would orders so much quantity etcetera.

You do not we do not we do not know what the level of stock will be that is something that will come up during that will something that will get realized once the demand itself gets realized right. But we can still plan for every possible level of stock that could get realized and that is essentially what we are doing here.

We are thinking of every possible level that could get realized and based on that we are trying to decide what should and for each such level we are trying to decide what the amount should be alright. So, this way of thinking about the problem when basically means that we what we are looking for is not just merely an action ok which is an action would be simply a specific quantity that you want to order, but rather an entire plan of actions.

So, the plan of actions is then a function its it which tells you what action you should be taking as if once you get a particular information right. So, what we are therefore, optimizing over is a; are functions; functions that will map your information which is in this case the amount of stock that you have at the time at the beginning of the time period to the action that you want to take alright.