Optimization from Fundamentals

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Lecture – 2C
Continuity and Weierstrass theorem

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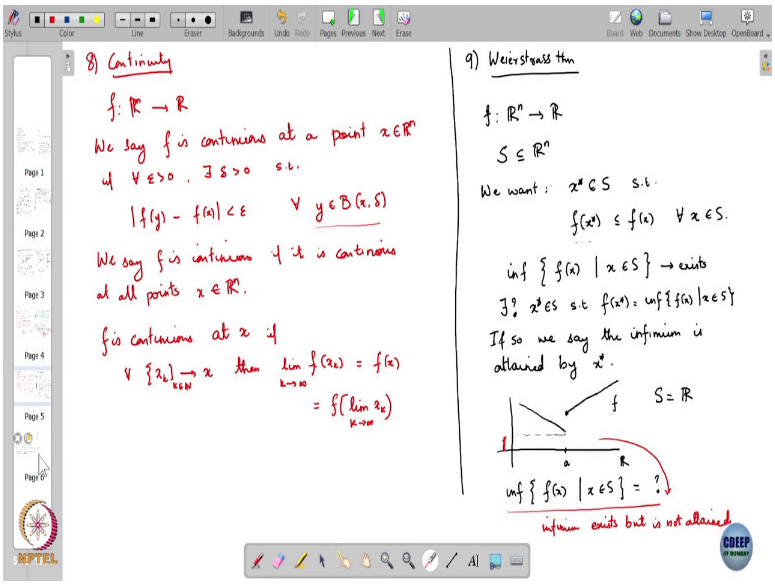

Let us go to the next one Continuity ok. So, function so let us look at this, let us think of, let f be this function and I will think of it as a function of n variables taking real values ok. It can also be a vector function it is not very hard to extend this to vector functions. So, it is a function for that takes from R n to R; that means, it is a function of n variable and eventual out value that it outputs is a real number right.

Now, we say f is continuous at a point x in R n if the following holds. If for every epsilon greater than 0, there exists a delta greater than 0 such that. So, we say f is continuous at a point x in R n.

If for every epsilon greater than 0, I can find a delta greater than 0. Such that, the difference f y minus f x if I look at it in a absolute value is no more than epsilon. For all y's that are within a delta radius ball of x. So, the challenge is thrown in this way, you tell you show you set how close you want your function value to be? That is your epsilon.

You throw me an epsilon and the challenge is that, I will be able to find for your delta. Such that if; I look at the function values in the delta radius ball around x ok. Look at y is that lie in a delta radius ball look at these y's that lie in a delta radius ball around x. Then the function values f of y are no more than plus minus epsilon from f of x ok, f y minus f x is utmost epsilon in absolute value.

If you can do this alright, then we say that the function is continuous at the point at the point x. So, we say the function is where f is continuous at x continuous simply. So, we say f is continuous if it is continuous at all points. So, there is another way of, now this is a complicate somewhat complicated definition. Because it is it is posed in terms of this kind of a challenge.

There is another way of there is another equivalent way of writing this, which is the following which is using the language of sequences. So, suppose if I have so we say that f is f is continuous. We say that f is continuous at x, if suppose you take if you take any sequence x k like this that converges to x. Take any sequence that converges to x, then along that sequence you look at f of x k. And look at the sequence f of x k its limit should be equal to f of x.

So, in the language of sequences, what the continuity definition is basically saying is you take any sequence of points x k in the in the in the domain of the function, domain space of the function. And look at the value, the sequence of values that gets generated in the range. Then that sequence of values itself is a convergent sequence, which converges to the value of the function at that.

So, another in effect, what we are saying is limit of f of x k is equal to f of limit of x k. So, the limit basically jumps inside the function, the evaluation of the function ok. This is what we mean by continuity ok. Now this then brings us back to optimization.

So, I had told you that, I had told you through the through Queen Dido’s problem. Why it is actually why we cannot take for granted the existence of a solution to an optimization problem? Because if a solution does not exist, you can you can get all kinds of absurd conclusions, just by arguing inductively without checking the existence.

So, we need a way of checking for the existence of a solution that does not entail finding a solution in the first place ok. Without finding a solution you should be able to be guaranteed, rest assured that the solution at least exists. Then you can go about finding it, if you want to right. So, that is what is guaranteed by this theorem of Weierstrass. So, for me to write Weierstrass theorem, you should first write out what it what we mean by optimization problem formally?

So, the way I stated the optimization problem previously was this z. We are we are looking for a you have a function like this, we have a function from R n to R suppose and you have a set S which is a subset of R n. And what we want: is a x star in S such that f of x star is less than equal to f of x, for all x in S, correct.

Now, in if effectively what we want is this, if you look at the set of values that f x over the set S. Look at the set of values that f x over the set S. So, f x such that x belongs to S look at this set. What we are what we want is effectively the infimum of that set. We want the least, we want the greatest lower bound on that set, because we because f of x star is the least possible value of f in this particular set, right.

But then that is the this is this is the, but when you think if you look if you if you think of it this way this creates a the furthering question. How do we know? We know that the infimum exists, for every bounded side and infimum exists thanks to completeness axiom. But the, but the completeness axiom does not tell us that the infimum is of the form f of x star for some point x star in the set. So, how do we know? So, this quantity exists as a real number.

But how do we know? Does there exists an x star in S, such that f of x star equals this particular S. It is clear. So, so these are two this brings us to two separate question one is when is the infimum itself finite, the second is when is there a matching when is when is that infimum actually achieved by a point in the set ok. So, if there exists such an x star we say the. If so we say the infimum is attained by x star, ok.

In that case it is both. So, if the infimum is not attained by x star then it is not meaningful to look for such an x star. It is not present in the set of alternatives that we are searching over, but it is meaningful nonetheless to look for the infimum of f nonetheless that is that that is meaningfully. Although the x star finding the x star itself may not be meaningful. Let me give you an example let us look at this function here.

I have this function on reals; so this is this is the domain of the function, where I am plotting its values. So, the function looks like this its decreases up till this point. Say suppose, point a at point a at point a there is a jump discontinuity, at point a its value is actually here not here. So, the value of the function at a is this bold dot here and then from here it increases suppose linearly like this.

So, now my this is my function here plotted out here, my S is just real numbers. So, can someone tell me, what is the, what is this?

Student: (( ))

What is this? You are tempted to say f of a, but then f of a is this.

Student: (( ))

And that is certainly not the minimum value of f.

Student: (( ))

Right. So, what is? So, so it is meaningful to talk. So, what is this particular thing? This particular thing is this is this point here, the point to which the function tends to from the left.

The value of the function to which it tends to from the left, but there is no point in the domain where it takes that particular value the best bet was a itself. But for a the value is up has been has is jumped up, there is no point in the domain for which this thing is attained right. So, this limiting value is attained. So, this is the case where as I said this quantity exists. This one exists is exists and is finite its some its some finite number. It is this height here, but there is no x star ok.

So, yeah; so it is a case where the infimum exists, but is not attained ok. So, this is equal to this side.

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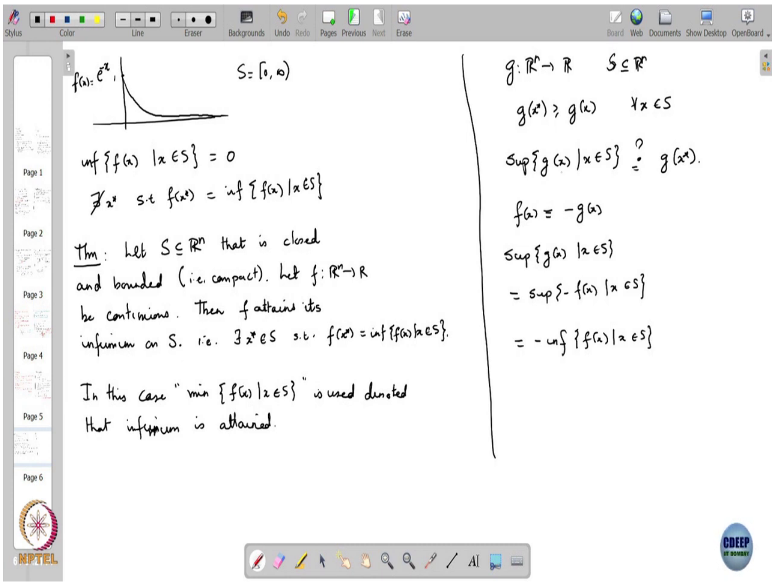

Let us take another example. Look at the function e to the minus x and the suppose S is 0 to infinity. So, e to the minus x how does this look? It at x equal to 0 this is going to be 1 and then it will decrease exponentially like this.

Now, what is again question here is again what is this? What is this infimum? 0.

Student: 0.

Right, after all that that is the greatest lower bound for all the function values of a. So, this is equal to 0, then what is the x star?

Student: (( ))

What is the x star? There is there is no there is no finite x star, I mean there is no there is no number there is no nothing that we call we can call as a real number which attains this value x, which gives you this particular infimum value.

So, there is again in this case that does not exist an x star such that f of x star equals this infimum. Now as you can see this is this is an issue that that you are faced with typically you would start with an optimization problem. You have a domain that that is specified in a in a sort of implicit way, you cannot enumerate every point and check what is happening in the domain? You have a function also whose terrain you complete you do not completely understand.

The function is specified in some sort of black box session. So, you do not know the full how the undulations are where these function is higher, where its lower, where it may have jumps or any of that. With that kind of knowledge, you now need to make sure that none of these kind of case is happen. That there is actually a that the that the infimum is in fact attained and there is it is meaningful to look for a solution x star for this, for the optimization problem right. So, that is what y stars theorem gives you. So, now I will state the theorem.

Let S be a subset of R n, that is closed and bounded in other words its it is compact. Let f be a function from R n to R that is continuous. Then f attains its infimum on S, that is there exists an x star in S such that f of x star equals infimum of f of x as x range is over here. So, there is such a thing called as in this case this case our notation changes, we are not then we are not looking at we do not call it the infimum.

In this case we when the infimum is attained we use the notation minimum ok, is the notation used. It is to denote that the infimum is attained, it is clear ok. Let me end with one small observation which I made last time also, but I will just make it more concretely. I have been talking about in optimization I have been talking about minimizing function, right. Here you are looking at out here you see, I am talking of f x star less than equal to f x.

So, I am looking at values that are lesser than all other values, but that is not. So, if you are if you are looking for, suppose you are you have a function g and you wanted to look for you wanted to find. Suppose g is a function from R n to R and you have some subset S of R n. And you are looking for a x star such that g of x star is greater than equal to g of x for all x in S. In this case the problem you are trying to solve is that of a supremum of this, supremum of g of x as x ranges over S and the question is whether this is attained by some g of x star, ok.

Now, what this that it is very easy to map this back to the setting that I the setting that we have considered which were where we are looking to minimize rather than maximize, ok. And that is by simply taking f x as minus g x identically equal to minus g x. If f x is minus g x, then the supremum of this is simply the supremum of minus f x or as x ranges or as which is it is easy to shows the same as my negative of the infimum of f x, right.

So, infimizing f is equivalent to supremizing infimum, right. So, as a result without loss of generality we will just always work with minimization because that also is very is some of it is very nicely with the convex shape of functions and convex sets and so on.