Optimization from Fundamentals
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Lecture – 2B
Open sets, closed sets and compact sets

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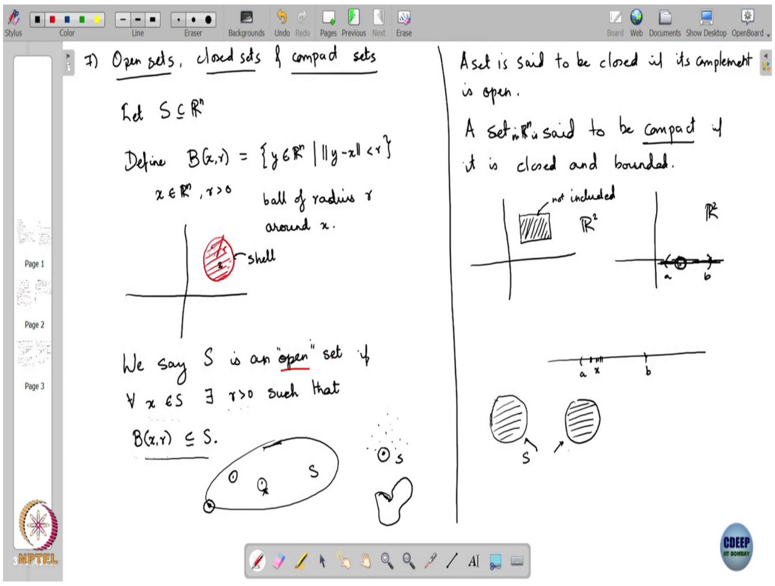

Now, let us define some more properties of sets. So, there are three different concepts that we will need open sets, closed sets, and compact sets ok. What is an open set? So, let S be a subset of R n ok.

And, define this B of x comma r, as set of y in R n ok. Where is so here x in belongs x is a point in R n, and r is just a real number, but it is just a positive number. B of x comma R is defined as all the y s such that, they are such that they are at distance strictly less than R from x.

So, here again my axis, I am just drawing 2 axis, here is my point x and if I take my r here defines a radius, and B of x comma r is then all these points. All of these points that lie in a ball of radius r around x this called a ball of radius r around x.

Now, because we are keeping this strictly less what we are excluding is those points that are at exactly distance r. So, the shell of this ball is actually excluded. So, if you want this shell here this one is this is excluded from this definition, from this set.

Now, we say that S is an open set; an "open" set, if for all x in S. So, take any point it has and around it. You should be able to find some ball ok, some ball, such that it lies completely in S ok.

For all x in S there exists a radius r such that if I look at this ball of radius r around centered at x this ball is completely in S. We say S is an open set, if for all x in S there exists an r such that, the ball centered at x of radius r lies completely in S ok. So, what is this such a set look like? So, just imagine a set like this.

So, what is if I take a point here, if I take a point x here; so, the challenge is that you have to for every point x in the set, I should be able to find a fit of ball around it such that it lies completely in the set right. So, let us look at a point like this, around this I can fit a ball very easily take this out of ball ok. I take a ball like this it fits very easily or inside the set, I take this point this ball also fit there is a ball around this that fits completely in it the difficulty starts arising at this point.

So, if this point which is here on the edge of the set. Now, what can I around this can I find a ball such that it lies completely in the set S. So, whatever ball I choose; however, small I make the ball, I will have a problem. Because, some of that some part of that ball is going to spill outside the set is just the way the set is right. No matter how small I make this, how tiny

I make the radius? Some part of the ball is going to spill outside the set and this condition here, this condition is going to get violated right.

So, sets like this, sets like this which have a shell or what we call a boundary those kind of sets are not open alright. Now, there are other kinds of sets now a set does not necessarily have to look like this like a contiguous body right. For example, a set can be just to isolate once have isolated points, this is also a set. Now is this set open, no the set is not open, the set is not open simply because you take any point around at any point in the set and try to draw this ball around it.

It has several other points apart from the points in the set right. So, the ball the ball has to be completely within the set that is not possible here ok. So, open set that is open will you can you should imagine it is as a set; as a set that is contiguous and has no shell ok. Now, that does not; that does not mean a set it does not mean that it should have a certain overall shape ok. For example, here I have drawn it almost like an ellipse.

A set could have we could have a shape like this also, some strange shape like this. And, if I exclude the shell around this that would be an open set alright. As an example take the ball itself, the ball itself the ball that of radius r around x that itself is an open set.

Because, every point I take inside the ball, I can find an even smaller ball that lies completely inside this particular ball right. And, I can this think a can I can keep doing this. Now complimentary definition to this is that of a closed set. A set is said to be closed if its complement.

So, it is said to be closed if its compliment is open ok. So, the way you should think of the closed set is precisely this it is basically you look at everything that is not in the set. And, check if that is an open set, then the set that you were you started with is a closed set.

A set is said to be compact, if now here remember I am talking of sets in r n there is a much more general definition outside of R n, it will for R n this is equivalent to all definitions a set is said to be compact. So, let me we more precise here set in R n is said to be compact if it is closed and bounded.

So, set is so, this sort of so, a compact set is a set that can be enclosed within some large balls alright. And, it has it is such that its complement is open right. The complement is open means the complement does not have a shell alright. If the compliment does not have a shell; that means, the set itself has the shell ok.

Now, the a set need not can be neither closed nor open right, it is these are not the only categories of sets. Of course, that it can be neither closed nor open say, for example, but whether a set is open or closed depends on this the ambient dimension in which we are looking at the set.

So, when I am in the definition of an open set, I am looking for I am asking that there should be around every point x, there should be a ball of radius r, there should be an r such that this ball, the ball B x comma r belongs to the set. Now, this B of x comma r is a ball defined in R n right. So, the set is asking for a ball in R n to be fit inside the set. So, I will give you an example for instance took a let us look at R 2 look at let us look at R 2.

And, look at this square excluding its boundary, excluding the boundary ok, this shell is not included. Then is this an open set? This is an open set. No problem every around every point I can find a ball that lies completely in the set. Now, let us look at R 2 itself and look at this segment here. Says it start which lies along the horizontal axis starts from some point a ends at some point b ok.

I have and let us suppose this is an open segment let me draw this better. So, it starts from some point a ends at some point b, but a and b are themselves not included ok. Now, is this set an open set?

Student: No.

So, what is the definition of open set in a demand? So, if I want to think of this as a set subset of R 2 right. I want to think of this as a subset of R 2, then I need to be able to fit two dimensional balls around every point in it such that the entire ball lies in the set. But, two dimensional balls will always spill outside the set right. So, as a subset of R 2, so, if I express this set using two coordinates and coordinate for the horizontal axis as well as one for the vertical axis. And, as a subset as a subset of R 2, this is not this is this set is not open.

However, geometrically the same object can be immersed also in R 1. And, the entire set can be expressed using only one coordinate. In that case, if I then that case the picture looks different, then, I do not have two dimensions, I am only in one dimension.

And, I am looking at something like this a and b excluded. And, now around every point what I need to look at look, what I need to fit is one dimensional ball right. A one dimensional ball is simply an open segment. And, a one dimensional ball can be always accommodated itself around every point alright.

So, whether set is open or not depends on what ambient space you have immersed it in ok. What technical the word the technical word is what ambient space you have embedded it in alright. So, it depends on the ambient. Being open or closed again has known has no relation to how many pieces the set is made of? For example, I can take an open ball like this another open ball.

And, their union is this and this. Let S be the union of these two together this region and this region together. If I take; if I take this as my definition of my side this is an open set, it does ok. It is made up of two is two pieces that is fine, but individually if you look at the pieces they are themselves are part of some sort of you know intuitively like a contiguous mass ok. So, this actually brings me to another point, which is what sort of operations preserve openness and closeness.

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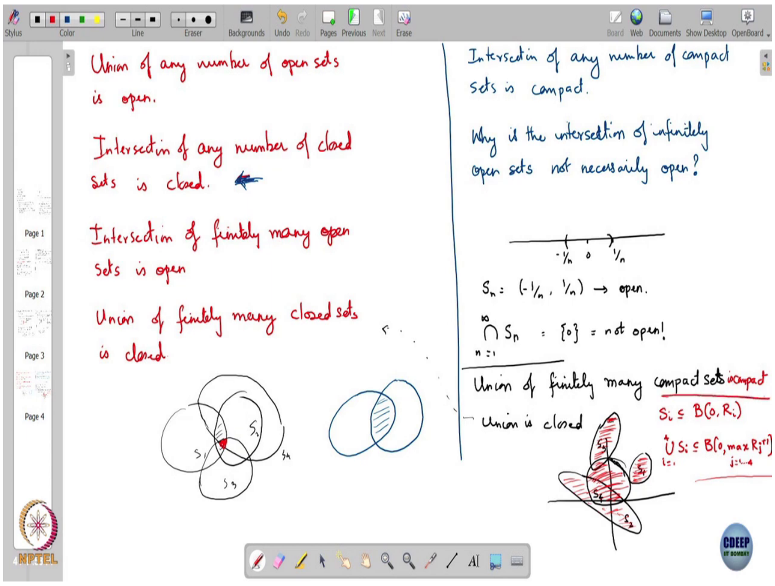

So, if you take the unions, union of any number of open sets is open you take any collection of open sets, take their union that the resulting set is always open this. So, by negation of this or by complementation you also get that the intersection of any number of closed sets is closed. The intersection of finitely many open sets is open and union of finitely many closed sets is closed. So, can someone tell me is the intersection of any number of compact sets compact any?

So, the intersection of any collection of sets; let us take this is one set S 1 take another set S 2 the intersection of these 2 is the common region between right. So, if I take the intersection of any number of sets it is the common area that belongs to all of them.

Another set S 3 here and another set S 4, S 4 like this etcetera. I can put together many of these sets right, the common area that belongs to all of them. So, here in this case the common area is turning out to be I think this red region here ok.

So, the common area of all of them is the intersection. So, what is the claim here, in the claim made here, claim made here is that the intersection of any number of closed sets is closed. Is the intersection of two bounded sets bounded?

Student: Set.

Or are any number of bounded sets bounded?

Student: Sets 3.

Why is it bounded?

Student: After 4 is the format.

It is part of thus any one of them for sure right. It is part of each of them right. Since, each of them is bounded that intersection is also going to be bounded right.

Student: Yes sir.

So, it is a so you are looking at this tiny region, which belongs to all of them. So, to claim that this tiny region can be put inside a ball, all I need to say is that any one of them can be put inside the ball, in which is what I have been told anyway. So, intersection of any number of compact sets will be bounded. And, this statement here was telling you that intersection of any number of closed sets is closed. A closed sets are already, so compact sets are already closed.

Student: Closed.

So, the intersection is also going to be closed and we know that the intersection is going to be bounded. So, this intersection is compact. So, intersection of any number of compact sets is compact. If the intersection of finitely many open sets is open. So, when you take if you take S 1 and S 2 as open sets in this particular figure. I have S 3 and S 4 here also. So, let me just draw this separately.

So, here is your S 1 is S 2, S 1 and S 2 have both in both in their definitions, I have excluded the shell around it. So, even in this even in the intersection, the shell these shells are not present. So, let us a so, the question here is this why is the intersection of infinitely many open sets not open. So, let me look, let us look at the so, let me write it here why is the intersection of infinitely many open sets. Technically, we should say not necessarily open not necessarily open.

Why is it? So, can we give an example?

Student: (Refer Time: 22:28).

So, let us look at, let us look at this example in the real line itself, here is my here is 0, here is minus 1 by n here is plus 1 by n. Look at this interval that starts from minus 1 by n to plus 1 by n excluding the endpoint. This is an open interval, it is an open set in r alright. Now, so, I define S n as this minus 1 by n plus 1 by n. Now, what is the intersection of all these S n's? n starting from say 1 till and going up to infinity.

Student: (Refer Time: 23:16).

The intersection of all of these is has only one element in it, which is the point 0 right. So, the intersection of all of these is this set. It is a singleton set having and this set is not open.

Because, any ball around 0 will have points that are positive as well as negative not just 0. Obviously, any ball of positive (Refer Time: 23:41) this is not open all of these were open.

Its fine let me ask you another question. What when about compact sets? So, we saw an intersection of any number of compact sets is compact, what about the union of finitely many compact sets? Union of finitely many compact sets, each compact set is both closed and bounded. So, they are all of them are individually closed, union of fine, they are finitely many of them, union of finitely many of closed set is a closed set. So, the union is closed; thanks to this statement right. So, the union is definitely closed. Now is the union bounded right?

Now, why is the union bounded? Let us look again look at this geometrically. Suppose, you had on my axis here has one set I have S 1. Here is some other set S 2, here is a third set S 3. Here is a fourth set S 4. You are looking at the union of all of this right. So, what is the reason I am looking at? The region is this use a different color, the region is everything that is there in S 3 and also S 4 and also S 2 and also S 1. This is my set. Now is this bounded?

Student: Yes.

Each of them were individually bounded right. So, S 1 could be fit inside.

Student: Yes.

A big ball of radius R, R 1 S 2 could put be fit inside a ball of radius R 2, S 3 could fit inside a ball of radius R 3, S 4 could fit inside a ball of radius R 4 right. So, each S i can be fit inside a ball of radius centered at the origin and of radius R i suppose right. Now, if all now all of them collectively can be put inside the largest of such ball of these balls.

So, these are R 1, R 2, R 3, R 4 they are concentric balls that take the largest of them, maybe make the ball a little bit larger and that will definitely fit all of them right. So, the union S i is certainly contained in the maximum of this. Just to be safe if you want to add a 1, you can add a 1 right. The union of all of these are contained in one.

So, the union of and this maximum here is a maximum of a few real number, it is a maximum of some finitely many real numbers, whatever it will be a finite number right. So, that so, consequently this ball is also a finite radius. So, the union is we just checked is closed we and we by this argument, we the union is now also bounded. So, the union is both closed and bounded. So, consequently the union of finitely many many compact sets union of finitely many compact sets is compact.