Optimization from Fundamentals
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Lecture - 14B
Proof of complementary slackness

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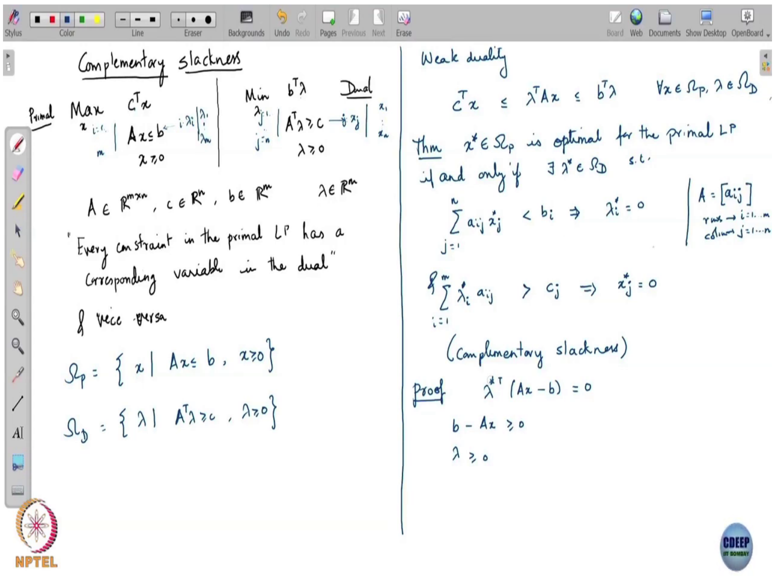

So, let us just do a proof of this. So, one convenient way of writing this particular condition is to simply write this lambda star transpose Ax minus b equal 0. Why is this, why can I write it like this? The reason is because I know that I know that, Ax minus b or rather b minus Ax is always greater than equal to 0. Since b minus Ax is greater than equal to 0, moreover lambda is also greater than equal to 0.

So, then their inner product will be 0, only if you have this; only if you have this sort of situation, where if there is slack in one of them then the other one must be, the other one must be 0, right. So, this would be a compact way of expressing this. So, let me write it more neatly.

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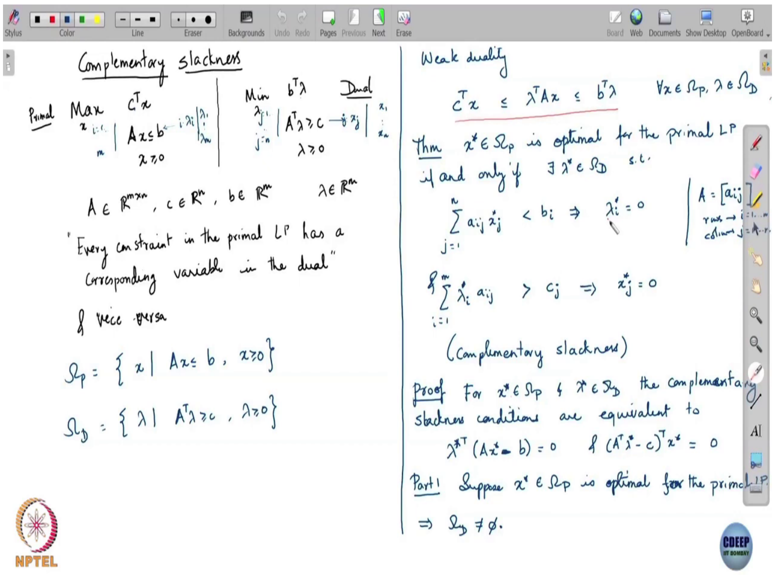

So, for x star in omega P and lambda star in omega D. The above condition; the complementary slackness conditions, [mentary] complementary slackness conditions are equivalent to lambda star transpose Ax minus b equal to zero and A transpose lambda minus c, the whole transpose x star, sorry lambda star here x star equals 0, ok, alright.

So, now we want to show the necessary and sufficiency of complementary slackness for the optimality of x star. So, part 1 is say suppose, x star in omega P is optimal for the primal LP. So, now if x star is optimal for the primal LP ok, is it possible that the dual is infeasible?

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It is not possible, because the duality theorem of linear programming we learnt that if the primal has a solution then, so does the dual right. So, it cannot be that; if you have a finite optimal solution for the primal, then you cannot have that the dual is infeasible.

So, which means that what this means is that omega D is not empty, ok. Moreover omega D is not empty and so it cannot be that dual is infeasible and moreover there always exists a solution to the dual. And the optimal values are equal. We know that from the theorem of from the strong duality theorem.

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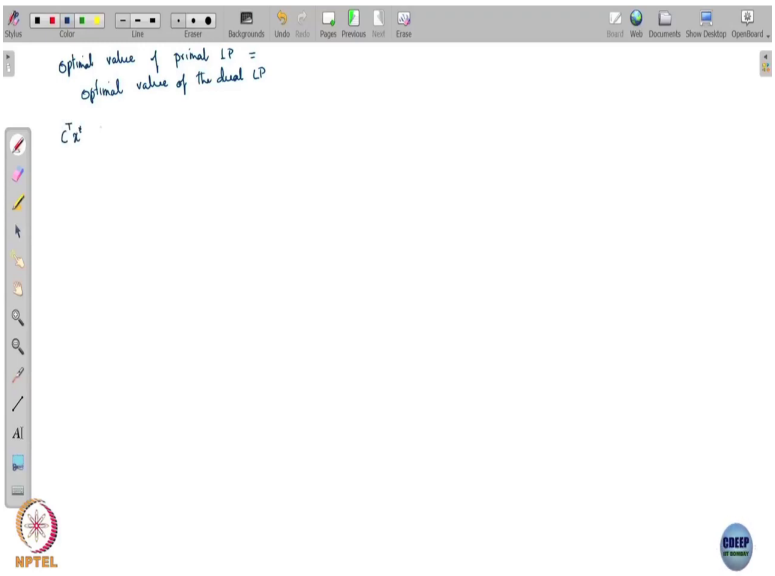

So, optimal value of primal LP equals the optimal value of dual. What does this mean? If I look at C transpose. so C which means that, first not only that the omega D is not empty, there exist a lambda star in omega D.

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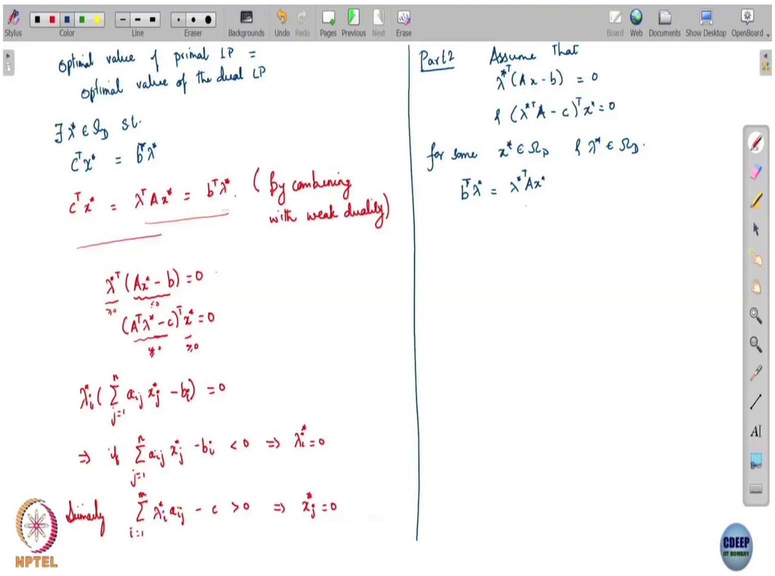

Such that C transpose x star, which was my primal optimal value is equal to b transpose lambda star, alright. Now, if we compare this with this particular, this strong duality statement compare that with the weak duality statement, which is written here, the weak duality statement which is written here. Then what say and this weak duality statement remember was for all x in omega P and all lambda in omega D.

So, consequently we get by taking x as x star and lambda as lambda star, what are we going to get? We are going to get that C transpose x star is equal to lambda transpose Ax star is equal to b transpose lambda star, by combining with weak duality, ok. So, now what does this say? What does this say this what does this; what does the red statement here say? Well, the red statement simply says that, now I can do the following, I can put I can take this part together there are actually two equations here.

There are there is one equation here and another equation here. So, let me take one of, each of them separately. So, what is, what this is effectively saying is that lambda star transpose Ax star minus b equals 0. And that A transpose lambda star minus c, the whole transpose x star equals 0, right.

And now, since this must since if these are if the if this has to be if these have to be equal to 0, what does this mean? This means that it has to be that component wise, component wise they should be equal to 0; that means, see remember this quantity is always less than equal to 0, this quantity is always greater than equal to 0. This quantity is always, this quantity is always less than equal to 0, sorry this is greater than equal to 0 and like and this is and this is also greater than equal to 0.

So, for the inner products of these vectors to turn out to be equal to 0, it has to be that component wise they are actually they are each 0. Otherwise, the you will not get that the inner product ends up 0. You in the first case you would get that the inner product was is negative, otherwise in second case you will get the inner product is strictly positive, right.

So, what does this mean? This means that lambda j, lambda star j times the times this, this sum that we had here, which is times this sum or lambda star i times this sum, which is give which is captured in this inequality. That must be so, lambda star i times summation over j equals 1 to n a ij x star j minus b equals 0, minus b i equals 0 which means that, what does this mean? If now, a ij x star j minus b i can be either equal to 0 or can or strictly less than 0.

These are the only two possibilities, which means that so if it is equal to 0 this holds trivially. If it is strictly less than 0 then the other the only way you can have an the product equal to 0 is that lambda star i itself is 0, right. So, that which means that, if a summation a ij, j equals 1 to n x star j minus b i is less than 0, must imply lambda star i equal to 0.

And similarly, we can show the other way around, the other similarly sum over, the sum if this inequality holds with strictly. Then the corresponding x star j must be equal to 0, right. So, this is one direction of the complementary slackness condition. What we have concluded so far is if x star is optimal, then it has to be that these complementary slackness conditions must hold.

Now, let us look at the other direction part 2; the other direction. So, assume that, so, assume that lambda star transpose A x minus b equal 0 and, lambda star transpose A minus c the whole transpose x star equal 0, ok. For some x star in omega P and lambda star in omega D.

Now, what does this, what does this say? Well, we can rearrange this a little bit so, the 1st equation simply says, b transpose lambda star equals lambda star a lambda star transpose Ax star. Now, the second one the second equation it is again says that. So, the second equation ok, I had made a mistake in writing this, sorry yeah.

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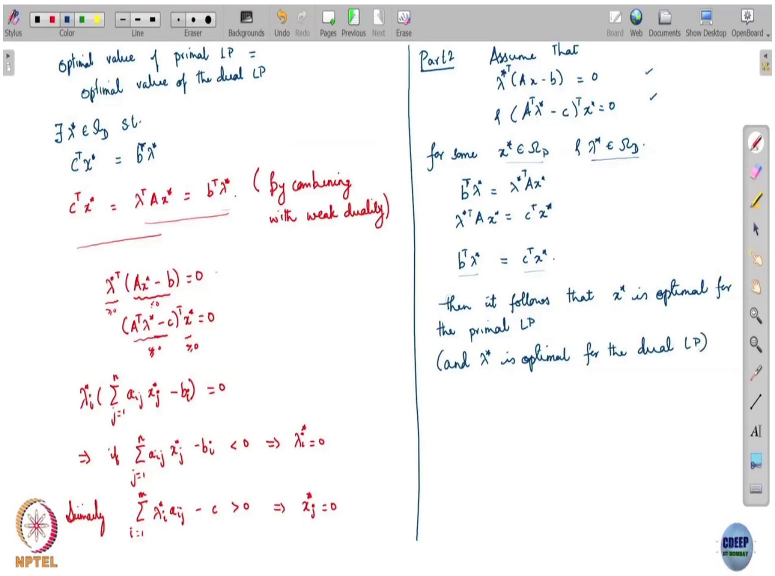

The second equation similarly says that, lambda star transpose A x star is equal to C transpose x star alright. So, now what does this mean, this means that b transpose lambda star is equal to C transpose x star. So, we verified that these two conditions hold for some x star in omega P and lambda star in omega D, from there we here we conclude that b star b transpose lambda star equals C transpose x star.

And now, what this what does this mean? We know from weak duality that, we know from weak duality that an inequality in this direction must hold and if any and if and we know from the strong duality theorem that, if equality holds then it has to be that these two are optimal in, ok. So, then it follows that x star is optimal for the primal LP and that lambda star is optimal for the dual LP, alright.

So, what does this? So, this completes the proof and what so, what have we learnt from this? We have learnt basically that as far as optimality is optimality of linear programs is concerned essentially it comes down to just ok. So, what does this theorem teach us? It basically teaches us that, if you have you if you take a candidate feasible solution x star for the primal and candidate feasible solution for the dual lambda star they are optimal if and only if they satisfy these two equations.

Now, these two equations if you see them they are actually not linear equations such that lambda star transpose something in x must be equal to 0 and likewise something in lambda transpose x star is equal to 0. So, this is the where the hardness of linear programming actually creeps in. So, the first the simply checking that x star and lambda star belong are feasible is a matter of checking that they satisfy bunch of linear inequalities and it is easy to generate solutions to linear inequalities.

But, you have to also for finding an optimal solution, you have to effectively end up solving some non-linear equation, even though the original problem was just the linear problem, ok. And this is the, this is the root of why this is linear programming is non-trivial. But, you will soon see that this kind of non-linearity comes up in all types of problems involving inequality constraints.

Student: (Refer Time: 16:46).

The non-linearity is in this is in this product equation. So, the lambda stars. So, sorry this was supposed to be x star the lambda star is multiplied with this to get you, and that product this quadratic thing has to be equal to 0, quadratic equation must be 0, likewise this quadratic equation must be 0. Yes, but the it is if you look at it as equations in your variables x star and lambda star, then these are not linear equations anymore, alright.

Another way of thinking about say expressing the same thing is that, if you look at this condition. This condition simply says that if something is true then something else is true. It is a conditional statement; it is not merely asking you to you cannot write this as simply a solution of linear equations. It says that if this is strict, then that must hold and if it is not strict with there is no particular, it says no it says nothing in particular right.

So, this so the so, this is the; this is why linear programming is actually harder than it appears. Because eventually you have it at its root, even though the original problem is just involves only linear formulations at its root to solve the problem, you are ending up making a solving some non-linear equation alright, ok.

So, that is one thing, but having said that the, if I gave you two candidate solutions to just verify that they are optimal is very easy. All I need to do is just check these two, check that they are feasible and simply check that this hold, right. So, finding one may be harder, but verifying is absolutely easy. Because all I have to do is just check this, right.

So, what this is done is taken a problem of which is of which is linear programming and reduced it to just simply checking some non-linear equations, checking for the satisfaction of some non-linear equation. Now, you will see why this is, why this is a you know a significant simplification?

Because we started off thinking of linear programming saying that all solution it must have a solution on an extreme point, but then there were so many possible extreme points, it was not easy to characterize them and we said, well if I gave you even one extreme point, how do I confirm that, it is in fact optimal without comparing with all the other extreme points of the linear program all of this is extremely hard to do. This is this in comparison is some you know something, that you can potentially do much more easily. Yes.

Students: Sir, are they actually (Refer Time: 19:42).

Not necessarily, not necessarily. So, the earliest algorithms for solving linear programs were actually went about their business by searching over extreme points, just cycled over extreme points and made sure that you are getting to a better extreme point at each step. And that is how they try to find the solution; modern algorithms for LP’s try to attack this directly try to solve these non-linear equations directly.

Students: (Refer Time: 20:12).

Yes, yes yes. So, effectively solving a linear program amounts to finding an x star, which is in the feasible region of the primal and alongside lambda star which is in the feasible region of the dual; such so, they; so, it means being feasible for the primal and being feasible for the dual, simply means that they must satisfy these linear.

They must lie in this polyhedra and, but in addition to that they must also satisfy these non-linear equations right, that is what, that is what it means to solve a linear program, alright. So, this particular condition the reason ok, I will also give you a bit of intuition on why this condition appears?

The reason for this is that if you look at the dual just remember I had told you that, I had mentioned this once. And this I will mention this again and later also that if you look at the dual variables, the dual variables are actually the same as what we were earlier calling Lagrange multipliers.

So, when we were looking at optimization problems with equality constraints, we had these additional variables, which we denoted by lambda 1 for each constraint. They came out of doing applying implicit function theorem on the constraint and so on. Now, those additional variables are actually the Lagrange multipliers.

Now, the if you look at the complementary slackness condition, what it effectively it says? The complementary slackness condition effectively says that, I need to have only Lagrange multipliers for those constraints that hold with equality. If so that although the constraint is stated as an inequality, if it holds with equality then there is an applicable Lagrange multiplier for that constraint.

If it does not hold with equality, which means if this inequality is strict, then the Lagrange multiplier for it is effectively 0, alright. So, that is if I that is what this, these conditions are saying. And it is the situation is completely symmetric between the primal and the dual the variables of the primal are the dual variables of the constraints of the dual or the Lagrange multipliers of the constraints of the dual ok, and likewise the variables of the dual are Lagrange multipliers of the constraints of the primal, alright.

And so you can make the same claim about the dual as well. So, if there is a constraint in the dual, that is strict then the corresponding Lagrange multiplier or equivalently the primal variable must be 0. Because it does not count effectively ok, you again as I said we will see this in, see this some more as we go further into a non-linear optimization.