Optimization from Fundamentals
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Lecture - 14A
Complementary Slackness

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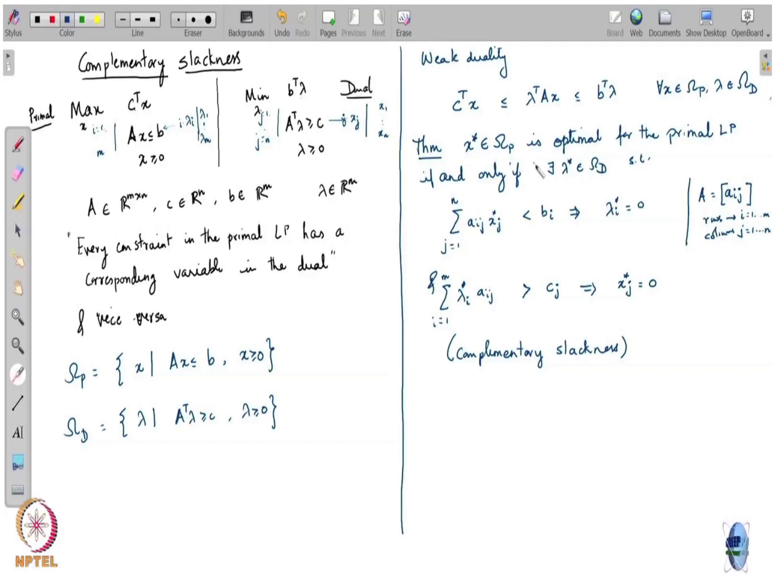

Ok. Welcome everyone. So, today, I will talk about an first to begin with I will talk about an important property called an important condition you can say called Complementary Slackness. And this is the condition that comes up because we whenever we consider optimization problems which have inequality constraints.

So, I will first illustrate this today in the context of linear programs, and then from there then we will take off to a more general problems. And hopefully, we will you will see either today or in the next lecture complementary slackness coming up as a condition in more general optimization problems.

So, let us look at this linear program. So, this is called let us say we are maximizing c transpose x subject to A x less than equal to b and x greater than equal to 0. Now, this is not in standard form, but and moreover it is also a maximization problem, but you can convert it to standard form, and then from the standard form derive it is dual.

And after you simplify the dual you will find that the dual takes this form. It is a min it is a minimization. So, here the maximization is over x. Your minimization, I will just use the variable lambda here. Minimization is over lambda of the objective b transpose lambda subject to the constraints that A transpose lambda is greater than equal to c and lambda is greater than equal to 0, ok.

So, now, let us repeat the observations that we had made about linear programs and their duals before as well. So, if my matrix A is in R m cross n, c is in R n and b is in R m, so in that case the number of inequality constraints that we have here these inequality constraints, apart from the constraint x greater than equal to 0, the number of inequality constraints that we have here this is equal to; this is equal to m which also happens to be the number of variables in this, in the dual problem, right.

The dual problem has m variables because lambda is a m length variable, is an m length vector, lambda is in R m. So, the number of inequality constraints that we have in the primal problem is equal to the number of variables of the dual problem. And if you look at the number of inequality constraints in the dual problem, that is equal to the number of variables in the primal problem.

And I had mentioned to you before that there is a close correspondence between these. So, what one can think of is that for every constraint in the primal, you actually have a variable in the dual and vice versa. So, every constraint in the primal LP has, so this is I have not written here, this is primal and this one is this is the dual. Every constraint in the primal has a corresponding variable in the dual and vice versa.

Vice versa means that every constraint in the primal has a corresponding variable in the dual, right. So, what I will do today is through this condition of complementary slackness I will actually make this more precise, and I will tell you how this which variable is actually corresponding to which constraint and so on, ok.

So, let us define a few quantities. So, let us define omega P as all x such that A x is less than equal to b, and x is greater than equal to 0. So, omega P is simply the feasible region of the primal and omega D is all lambda such that A transpose lambda is greater than equal to c and lambda is greater than equal to 0.

Now, because these are duals to each other these two problems, the we have you can easily verify that we already have weak duality. Weak duality between them, what does this mean? If I take any x in that is feasible for the primal and look at c transpose x, and that has its value is less than equal to, ok. So, what have I done here? What I have taken? I have, so this is true for all x in omega P and all lambda in omega D.

So, what have I done to get to this relation? I have c transpose x. So, and I know that c is less than equal to a transpose lambda or c in other words c transpose is less than equal to lambda transpose A. So, I am multiplying both sides by the vector x and x being greater than equal to 0; because x belongs to omega P, x is greater than equal to 0, then that ensures that the inequality is not flipped.

So, in short, I am taking the dual, I am taking the dual constraint and then multiplying both sides by x that gives me lambda transpose A x is greater than equal to c transpose x. But then, lambda transpose A x is actually the same as it can be it can be further bounded because A x is less than equal to b.

So, that can be further bounded and then you get that it is bounded by b transpose lambda, ok. And again, lambda being greater than equal to 0, let me make sure that my inequality is preserved. So, this is like we did in the standard form, you can get you have weak duality here as well, right.

Now, the important thing here is your the complementary slackness condition and, so let me write the theorem now. So, x star in omega P is optimal for the primal LP if and only if there exist a lambda star in dual in omega D, I should write capital omega D such that now, such that let me write it like this.

So, suppose let me on the side let me introduce a bit of notation. So, let the if the matrix A that is represented as a matrix of numbers a ij, where i runs from 1 to m and j runs from 1 to n. So, i corresponds to rows, and j corresponds to columns, ok. So, x star in omega P is optimal for the primal LP if and only if there exists an lambda star in omega D.

That means, it is a variable that is feasible for the dual LP such that you have the following two conditions hold. If I look at a ij, x star j and I sum this from j equals 1 to n, this quantity is less than b i and this implies the condition that this is the condition that this inequality holds means that summation a ij x star j from j equal to 1 to n is less than b i, this should this can this implies lambda star i equal 0, ok.

And the other way around as well. The other condition is that if I do a again do is a ij and write it this way. So, they consider the summation lambda star i, a ij, i running from 1 to m, that is less than c j implies x star j equals 0. So, when so what this means is, so, if the first inequality if this inequality here the way you should understand this is it says that if this inequality here is strict. So, then you must have that lambda star i corresponding to that inequality is 0, ok.

So, for every so, what is this inequality? This inequality is the ith constraint of the primal LP. This is the ith constraint of the primal LP. So, if the ith constraint of the primal LP holds strictly, ok means it does not hold with equality then the lambda i corresponding to that must be 0. Likewise, if you look at the jth constraint in the dual LP, if that holds strictly, ok. Sorry, this there is a, there is slight mistake here in the direction of the inequality. So, sorry there is a mistake in the direction of the inequality I just corrected it.

So, if likewise, if this inequality holds strictly, the jth inequality in the dual if that holds strictly then the corresponding primal variable x j must equal 0, ok. So, what you should; then the way to think about primal and dual variables is that what you have. So, your lambdas are actually the lambda, the variable lambda i which multiplies with b i in the objective is the one that corresponds to the ith constraint here, for the ith constraint here you have a variable lambda i.

And likewise the variable c j x j that multiplies with c j in the objective of the primal is the variable that corresponds to is the variable that corresponds to the jth constraint in the dual. So, what you have here are actually there are from i equal to 1 till all the way till m, there are m of these constraints, and then for each of these constraints you have a you have a dual variable which is lambda 1 till lambda m.

The, and likewise in the dual you have constraints now going from j equal to 1 all the way till j equal to n and corresponding to each of those variables you have dual variables of the dual which are actually the primal variables. So, those are then x 1 till x n, ok and these variables when you have when you are at optimality, they end up satisfying these constraint, these conditions. This is equivalent to being optimal.

That means, whenever the ith constraint in the primal LP holds with strict inequality the corresponding dual variable must be 0 and whenever the jth constraint in the dual LP holds with strict inequality the corresponding dual variable of the dual which means the primal variable x star j must be equal to 0, ok. So, this; so we will, this is what is called Complementary slackness.

So, what this is referring to is that whenever there is whenever there is the word complementary slackness only says this that whenever there is slack in one of these constraints, then the there can be no slack for the dual variable corresponding to them. Means the dual variable must be at its least value. Likewise, if there is a slack in this constraint then this one must be at its (Refer Time: 15:47) well, ok.

Student: Sir.

Yes.

Student: Is lambda i also (Refer Time: 15:53) value?

Yes. So, it will turn out that moreover it will turn out that lambda star is; so, that is a that is a good point. So, x here this statement only says that x star is a optimal for the primal LP if and only if there exist a lambda star like this. It will turn out that lambda star itself is actually optimal for the dual LP, ok. So, in fact, x star and lambda star end up being optimal for their respective problems through these conditions.