Optimization from Fundamentals
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Lecture: 2A
Bolzano-Weierstrass theorem and completeness axiom

Ok welcome everyone, we will continue with our topic of Optimization. If you remember in we were previously studying what is called real analysis.

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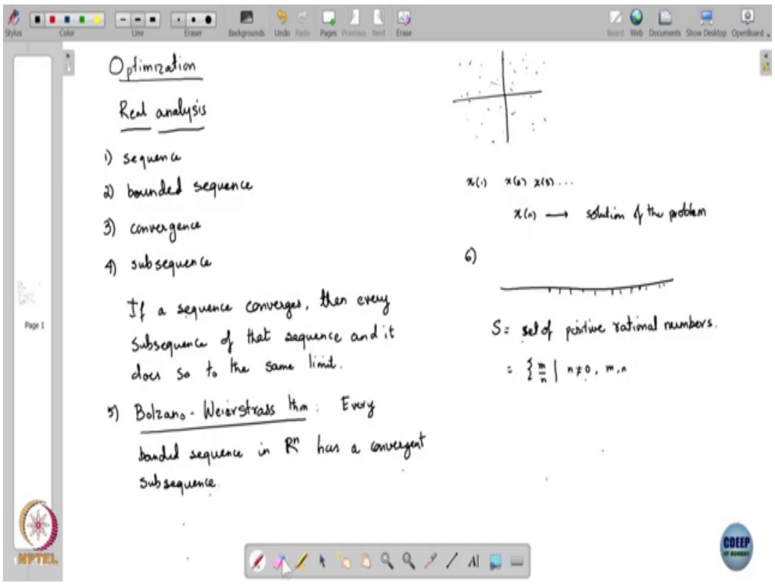

Real analysis is the careful study of real numbers and real numbers and their associated properties. So, we studied the following topics so far; we defined what a sequence is. We defined what it meant for a sequence to be bounded, so we defined a bounded sequence.

We defined what it means for a sequence to converge, so all concepts related to convergence what it means for a sequence to converge what it means for a sequence to not converge and what it what the limit of a sequence is and from that we also defined what is called a sub sequence.

These were the 4 topics we covered so far in real analysis, now let me take off from here. So, it is very easy to see that if a sequence converges then every sub sequence of that sequence converges, then every sub sequence of that sequence converges and it converges to the same limit it does so to the same limit as the original sequence right. now can you tell me if the converse of this statement is true? So, if a sequence if every sub sequence of a sequence converges then does the original sequence itself converge yes no.

Students: Yes sir.

Yes ok

Student: (Refer Time: 02:39)

Someone saying yes why yes.

Student: Sequence itself is the certainly sub sequence.

Ok, that is the correct answer. So, the sequence itself is the trivial sub sequence of the original sequence. So, the converse is actually trivially true it this statement is the non trivial one alright. Now let we were so far looking at sequences that were in living in R n, so these were vector the sequence of vectors.

Now let us come down to sequences that are in on the real line only ok. So, in that we have a famous theorem which is called the Bolzano Weierstrass theorem Bolzano. Bolzano Weierstrass theorem simply says the following, that every this does not necessarily have to be in a this is not necessarily true for real numbers, so I will let make this is true for vectors also.

So, I will tell you a more general version of the same statement. So, the simply the statement is simply this, that every bounded sequence in R n has a convergent sub sequence. You can see the sequences in R n and make look at those if you and consider is a bounded sequence in R n and look at all its possible sub sequences.

Amongst all its possible sub sequences you will find at least one which is convergent to some limit ok, it cannot happen that all the all the sub sequences do not converge and yet the sequence remain bounded it is not possible alright. Now the this is actually a very deep fact essentially what it is saying is that if you constrain the sequence to be in a box.

So, if you suppose here are my axis and my sequence is living in this bounded box, eventually it will the very fact that you are constraining it to live in a bounded box would mean that you can extract some sub sequence out of this such that that sub sequence eventually starts accumulating around some point.

The sequence cannot dance around in in such a wild manner that even its sub sequences are always dancing around and never will you find a single sub sequence which is converging ok. So, this is actually a very useful fact because it may convergence of algorithms are make extensive use of this particular fact proof convergence proofs of algorithms.

I will just give you an; give you a general sense about this, see when we are computing something in optimization. What we are effectively generating is a sequence of point sequence of iterate sequence, you have iterate at one then another you go through another loop get to the next iterate then you get to the next iterate then you get to the next iterate etcetera.

So, what you get are these is this iteration x of 1 x of 2 x of 3 etcetera etcetera etcetera and what we want is eventually that this sequence. If you look at the limit of this sequence x of n, this limit should converge to the solution of your problem solution of the problem under consideration right.

So for example, if you are looking at this so if you want to if your it means we are talking of optimization problems you want to this to converge to the solution of your optimization problem.

Now to show one of the big challenges that occurs in analysis of algorithm this making sure that this actually happens and you will be often the steps towards arguing that go through the argue go through you need first a way of arguing that the sequence converges at all to begin; with and that comes from Bolzano Weierstrass theorem.

You argue that there is at least a sub sequence that converges and from there you build your argument, this is a common way of argument alright. So, this is Bolzano Weierstrass theorem. Now let me come to sequences or sets in the real's ok. Now let us look at just the real line and look at suppose some set of points on the real line ok.

So, for instance look at the set which is the let me I will just write this in words the set of positive rational numbers for instance. So, this is the set of all numbers m by n, n not equal to 0 m and n sorry, m comma n both positive and they are natural number ok.

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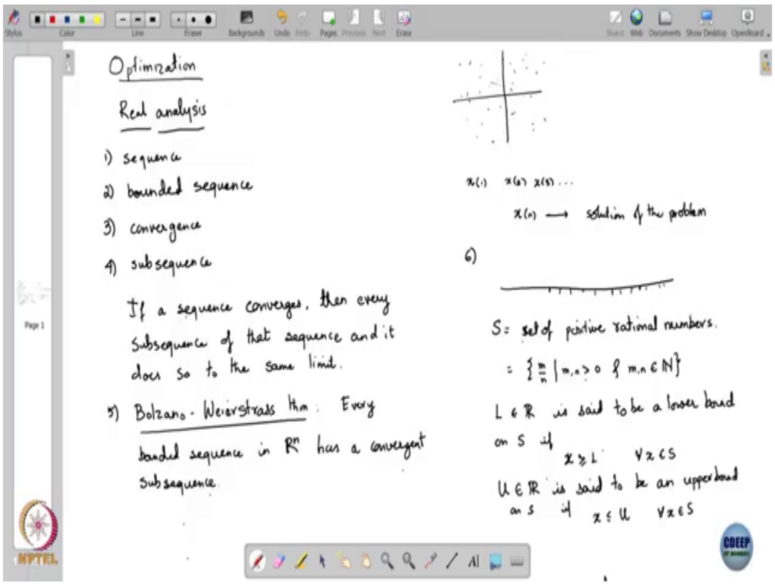

So, if you look if you look at this set this is a set of all positive rational numbers ok. Now we if I we say that a number L is said to be a lower bound L is said to be a lower bound if it is less than equal to all elements of S.

So now, if you take S to be the set of positive rational numbers what can you give me an example of a lower bound? Minus 1 0 these are all lower bounds. Likewise number u in R is said to be an upper bound on S if the opposite is true x is less than equal to u for all x in S ok.

Now, again for the same example set of all positive rational numbers can you give me an example of an upper bound there is no upper bound ok.

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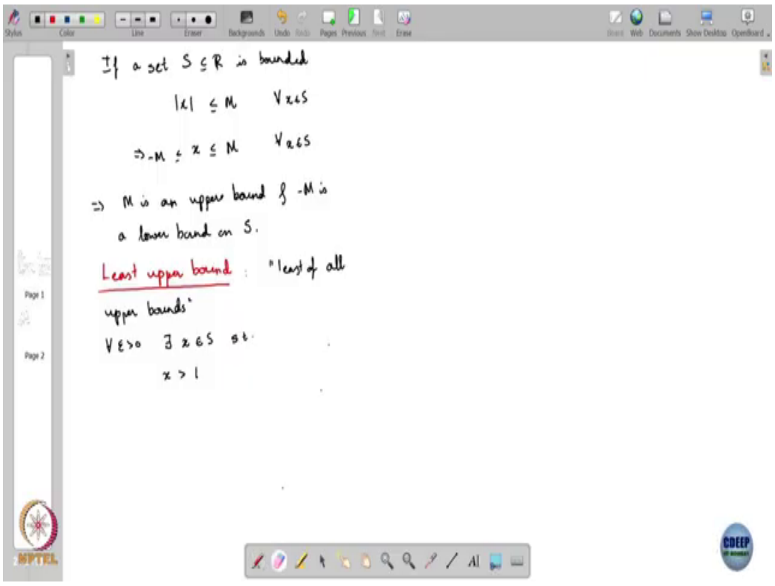

Now, what we will so if a set is bounded if a set which is subset of R is bounded then does it have an upper bound and a lower bound?

Student: Yes.

By definition right so it does have an upper and lower bound, why what did we say what did we mean by a set is bounded? We said we look at the norm of the of the elements in that set and all elements should have a norm less than equal to a prescribed number right. So, what it in this case we are talking of set of real numbers.

So, the norm will just the absolute value of the number. So, the absolute value of every real every number in that set is between minus m to plus m; where m is some m is some positive number right. So, if the absolute value is between minus m to plus m then it what does this mean? So, mod x is less than equal to M, this means that x is less than equal to M and greater than equal to minus M.

What does this mean? For all x in S for all x in S. What this means is that M is an upper bound and minus M is a lower bound is clear. So, a set if it is bounded will have an upper bound as well as a lower bound. Now upper and lower bounds are not unique; if you if M is an upper bound then M plus 1 is also an upper bound ok. Likewise if L is a lower bound then L minus 1 is also a lower bound. You can keep making lower bound smaller and smaller they will continue to be lower bounds. So, you can make upper bound larger and larger they become continue to be upper bound.

So, here is the another concept, which is called the Least upper bound, least upper bound is the smallest such upper bound amongst all upper bound ok. What does this mean? That if I take a; if I take a number even slightly smaller than this least upper bound then it cannot be an upper bound ok. This let me write this at English first the least of all upper bounds, what does this mean? If I for all epsilon greater than 0 there exists an x in S such that x is greater than sorry sorry just write this better.

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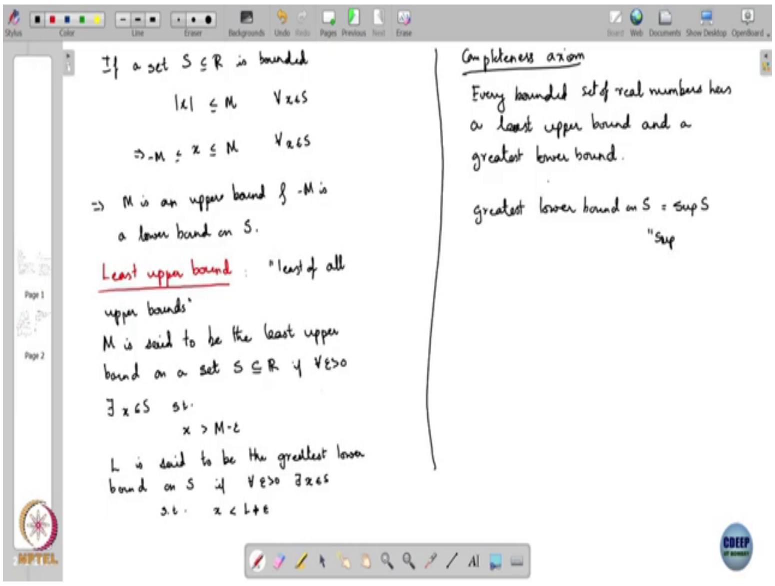

M is said to be the least upper bound on a set S set of R, if for all epsilon greater than 0 there exists an x in S such that x is greater than; x is greater than M minus epsilon.

So, M is said to be the least upper bound on a set in S sub which is a sub set of real numbers, if for every epsilon greater than 0 we can find an x; such that x is greater than M minus epsilon right. So, you cannot make M even slightly smaller and still have it as a lower as an upper bound, because there will be at least one element whose value will be greater than M minus epsilon for every epsilon ok.

Likewise L is said to be the greatest lower bound on S if for all epsilon greater than 0; there exists an x in the set such that. So, if I increase L even slightly, then I can find an element that violates my axiom is this fine? Ok.

Now, there is a chicken and egg problem that arises in trying to define what this trying to show that such a thing exists. The very fact that you show if you want to show that the a set must have a least upper bound, then what you want to what you need is that the set of upper bound must have a lower bound.

Likewise if you want to show that the a set must have a greatest lower bound then it means that the set of lower bounds must have an upper bound a smallest such upper bound right. So, that so the this is what I am using what is called the completeness axiom. Completeness axiom simply says that the every bounded set of real numbers has a least upper bound and a greatest lower bound.

So, if your set is bounded then it does have a least upper bound and greatest lower bound. If your set is just bounded on one side means; its bounded above then it has a greatest it has a least upper bound, but may not have a, but does not have a greatest lower bound.

If it is bounded below then it has a greatest lower bound, but does not have a least upper bound, but if its bounded it has both least upper bound and greatest lower bound ok. So, this leads us to the definition. So now, so once you with the axiom you know that such a thing exist. So, the greatest lower bound has a name instead of calling it a greatest lower bound on S. We simply we simply say the supremum of S and it is written as this sorry great I mean my mistake here.

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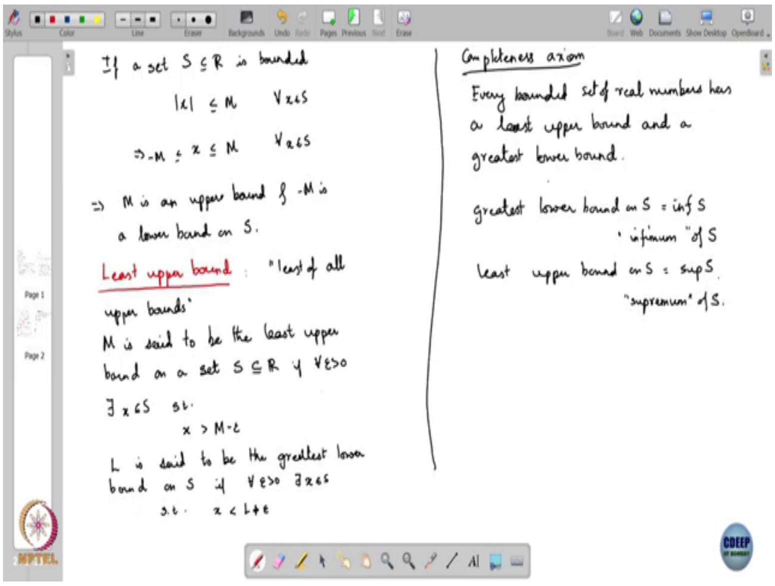

We simply say the infimum of S and the least upper bound is called the supremum ok.