Optimization from Fundamentals
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Lecture - 13B
Max-flow Min-cut problem

Let me give you another application, which is again a very popular application and also very revealing one. So, this is a, this is the famous Max-flow Min-cut theorem.

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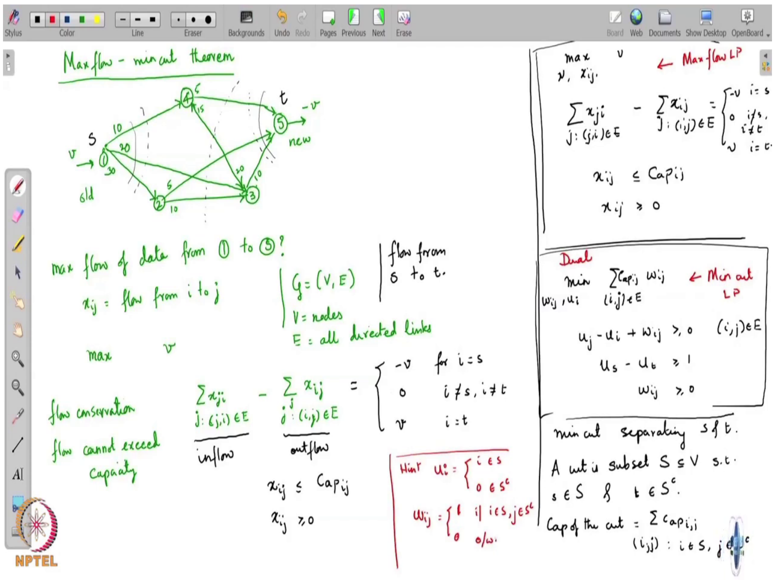

So, what is this; what is this theorem about? Let, so essentially just imagine you have a network say, let us let me draw one network here comprising of just five nodes, ok. Node 1, node 2, this is node 3, node 4, this is node 5.

And I can the there are links in this links or edges in this network and these are directed edges. So, means they denote what direction you can travel on this link, some links are bidirectional. So, like this 3 to 4 you can travel from in either direction, ok.

And now these links all have capacities; capacities means, they say, it tells you how much flow or how much material or how much data whatever can flow on this on that particular link, ok. So, let me just write them here. So, suppose capacity from 1 to 4 is say 10, capacity from 1 to 3 is 20, capacity from 1 to 20 1 to 2 is 30, say 2 to 5 is say 5; this is suppose 10 again, this is 15 suppose, here. So, from 4 to 3 is 15, from 3 to 4 is 20 suppose etcetera.

So, I have just put in some numbers here, from 4 to 5 is 5 again etcetera, right. So, suppose this is 10, ok. So, this is what the. So, what this means is this, the setting is as follows; you have this network of nodes, this network you can imagine as a network of say computers and you have data at say computer 1 here, this is where you have data.

And you would want that data to eventually reach computer 5, ok. So, suppose you have you are changing your mobile phone; you have your old mobile phone and you have your new mobile phone, you want to transfer data from your old mobile phone to the new mobile phone, ok.

So, the old one is node 1, the new one is node 5 and you have various possible means by which you can transfer this data. So, you can for example, move some, you there is one means which is to go through this intermediate things; if go through say Wi-Fi, a Wi-Fi connection that will connect you to 2, then take you to 3 and then it will go to 5 or from 2 it will go directly to 5 to your new mobile phone.

Or maybe you can use near field transfer to that could be another means; you have you could have another third mean which, in which you can copy or some of it onto a onto say a USB disk and then move that back to the new phone etcetera. These are all basically different paths by which data can potentially go from your old phone ok, your old phone to the new phone, ok.

And the question asked is, what is the maximum data flow you can get, ok. What is the maximum flow of data or how many MB's per second for instance from 1 to 5? Now, you can see what the complication here is. So, if you look at 1 here, flow is going to originate from 1; it is going to go out of 1, it the links that go out of 1, they have capacities 10, 20, 30. So, you may think that well on the face of it, you may think that you can send 60 MB's per second, suppose 6 these are suppose MB's per second.

So, 10 MB per second here, 20 MB per second here, 30 MB per second; you may think that, that is what you can send that is what you can send out of node 1. But then that data will later have to pass through this network and where the links are themselves, where it will depend on how what the link capacities are there are.

The link capacity is downstream could be very small; like for example, this link from 4 to 5 has a capacity of just 5. So, even though you are pumping in 60 MB per second here; it may not actually go, the links downstream may not be able to support that kind of data flow right, because they do not have that much capacity.

So, for so when whatever data reaches 4, only 5 MB per second can go to 5 along this link right; that is a limitation, similarly only 15 can go from 4 to 3, right. So, you can say basically the outflow from 4 is at most 15 plus 5 which is 20, right yes. So, one of the assumptions we will make is that, there is no storage of data; in the sense that this is all whatever comes in has to go out at the time scales that we are talking about, whatever comes in goes up, ok.

So, there is no these links are not storing data, ok. So, in that case, so the question really is of; if you want to look at, if you want to know what is the maximum flow of data from 1 to 5, you the answer to that comes down to somehow trying to figure out what the bottleneck is in this entire network, right.

Where in this network are you going to encounter the bottleneck? Means that where you cannot, what is the really the, what is it that is throttling your the flow of your data, right. And that is basically what this the this particular result actually tells you. So, now this problem of maximum of finding the maximum flow from 1 to 5, we can write this as a linear program, ok.

So, let us just look, let us see how we can write this as a linear program. So, for simplicity, let me suppose v is the flow that comes into 1 here and so, v would also be the flow that goes out of 5 ok; because now data is stored that is the; that is the flow that we will get out of 5. So, my convention would be that, any inflow would. So, x ij will denote flow from i to j ok and v is going to denote an inflow. So, at node 5, v will be an outflow.

So, we will denote it as minus v; minus v is therefore the outflow, ok alright. So, what is the objective then? The objective is to maximize this value v; v which is maximize the value of v subject to what constraints?

What are the constraints for us? The constraints are that we have flow, there are two types of constraints. The first is a flow conservation constraint, whatever comes in must go out right, whatever comes into any node must go out of that node.

Student: Yes.

So, that is a flow conservation constraint, flow conservation. And the second is the constraint that you cannot exceed the capacity of a; the flow on any link cannot exceed the capacity of that particular link ok, flow cannot exceed capacity. So, what is the, let us write out flow conservation; we do not need to write it for this particular network that I have drawn, let us write it more generally for any network.

So, my network let me denote it as a graph which vertices V and edges E. So, V is my are my vertices or nodes, edges E are these E consist of all the edges, all directed edges, directed links or edges. So, the flow conservation essentially says, if I look at x j i. So, and i sum this over j such that j comma i is an edge or a link in the graph.

What is this referring to? What is this quantity in terms of my notation? My notation was x ij is the flow from i to j, right. So, what is x j i? This is the flow from j to i, right. So, as I and I sum this over all j which have a link incoming link to node i; this is the inflow into node i, right.

So, this is inflow into node i minus, let us do this summation over j again; x ij, j such that now i comma j is a link in the network. So, this is all the outflow from node i; this is the outflow from node i, ok. So, this is outflow and this is inflow, ok. Now, what should this be equal to? Now, if you are at, if you are consider node 1; let us say, let us suppose I want to transfer data from a nodes, you want to transfer or flow from flow, if you want to get flow from a node s to a node t, ok.

So, at node s which is your origin node, at node s this right hand side would be equal to; what would this right hand side be equal to? This is the net flow in terms. So, at node s, there will be there is the inflow is only this value v, right. So, this will be equal to therefore, minus v for i equal to s; at every other node, the inflow and outflow should be equal.

So, this would be equal to 0 for i not equal to s and i not equal to t. And that node; and that node t which is where you want the flow to end up, at node t this right hand side would be equal to v, ok.

So, with my fictitious flow here v added here; you can see this is I can always write my flow conservation in this way. So, v is the flow that is coming in, all these are the flows that are going out; likewise minus v is the flow that goes out and these are all the flows that come in, ok.

Then I have my constraint that my flow cannot exceed capacity, in the network I cannot exceed the capacity. So, this x i j has to be less than equal to the capacity of link ij. And finally, x i j has to be greater than equal to 0. So, you can see this gives me the following linear program, let me just write it neatly here ok; you have you are maximizing v over variables v and all these x i j's.

Student: Subject to.

Subject to the constraint that subject to these constraints. x j i sum over j, where j i is an edge minus x i j sum over j where i j is an edge is equal to v for i. So, equal to minus v for i equal to s, 0 for i equal to s, i not equal to s comma i not equal to t. And v for i equal to t and x i j less than equal to the capacity of link i j and x i j greater than equal to 0. So, this is now, this is my, the solution of this problem will give me the maximum flow that I can send from node s to node t, right.

Now, if you write out the dual of this problem; it turns out that, the dual takes this particular form. So, if you write out the dual of this problem, it takes this form; see the dual now it has variables w i j, one for each link and in addition to that, it has these variables u i, u i one for each node.

And the way they are written is and the. So, the and the objective is this c summation c i j, not c i j; let us write it as cap i j, cap i j times w i j, ok. So, capacity of link i j times w i j is the objective. Subject to the constraint that u j minus u i plus w i j is greater than equal to 0, for all links i j; u s minus u t is greater than equal to 1, w i j is greater than equal to 0, ok.

So, this becomes the dual of this of the problem that we have written above, ok. So, you take your, you take the Max flow problem which write that as a LP; now this LP is not in standard form, but you can convert it to standard form as a minimization LP in standard form.

Then use the standard form dual and then you will get back to this particular problem; this is how you get to the dual of this of the earlier of the max flow problem. And if you analyze this it, can you tell me what would this amount to? What would be the solution of this LP, of the dual LP?

Student: (Refer Time: 17:47).

So, if the solution of this dual LP is actually has a beautiful interpretation; it gives you what is called the minimum cut ok, the minimum cut separating s and t. So, let me explain what that means. So, remember what I told you about the maximum flow problem.

The maximum flow problem is essentially comes down to finding the bottleneck in the network right, essentially if you are if you have an understanding of the bottleneck in the network, like how much that gives you a sense of how much flow it can potentially set, you can potentially set.

But if you look at the way we have posed this problem; the way they max flow problem has been posed, there is no mention at all of any bottlenecks or anything like that, right. But you can, but it the way that manifests itself is actually through the dual. So, what is the, what is a cut separating s and t? So, let me explain that. So, if suppose this is my node 1 is my source node, t is my destination node which is 5; a cut separating these two is just is a subset.

So, what I do is, I just do; I can cut the graph in this sort of way, in such a way that the vertices are divided into two sets, one set that contains my source node, the complement contains my this destination node.

So, if I take my all my. So, a cut is a subset S of my vertices or my nodes, such that the source node lies in S and the destination node lies in S complement. So, look at look. So, what is a cut here; for example, this is a cut, if I say for example, I can take 1, 4 and 2, the vertices 1, 4 and 2 that forms a cut.

So, 1, 4, 2 are has my source node 1 and then its complement which is 3 and 5 has my destination node. But I can create other cuts also; like for example, I can create a cut like this, I can keep only 5 in my in the on the destination side. So, I have my cut can be 1, 2, 3 and 4. So, 1, 2, 3, 4 can be a cut; because that contains my source and 5 which is the complement that contains my destination.

So, that is also a cut. Another cut is was is this one, where I keep only one in my set s and then 2, 3, 4 and 5 are all in the destination set on the that contains my destination node t. So, these are all possible different cuts, right. So, the min cut; so the. So, if I ask you what is the minimum cut, minimum capacity of a cut what that means is, look at the edges that are going across the cut; that means edges that have one node in one set and the other node in the other set, ok.

So, the capacity of the cut equal to is the sum of the capacities of i j, over all i j such that i belongs to s and j belongs to s complement. So, look at those nodes, look at those edges rather whose that originate in i, whose the left endpoint is in i and the right endpoint left endpoint is in S and right endpoint is in S complement.

Look at the capacity of those; look at the capacity of those edges and see all such edges; the total capacity of that is called the capacity of the cut.

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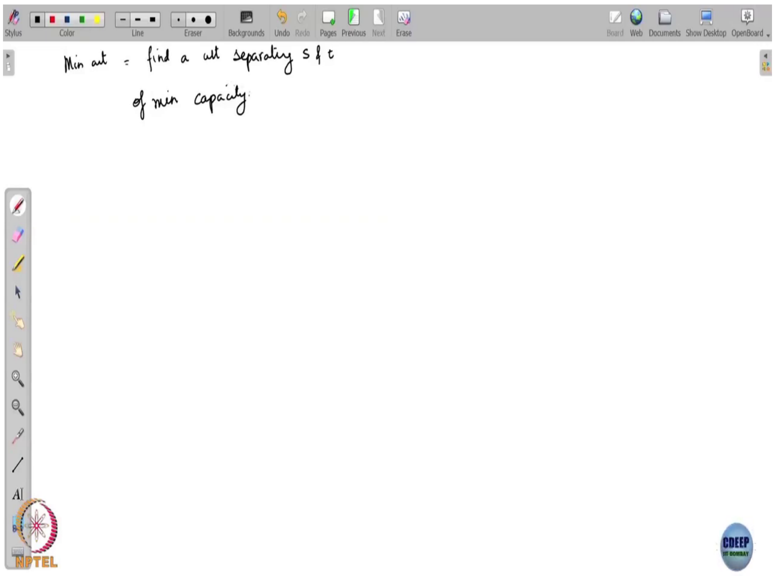

The problem of minimum cut is find a cut separating s and t of minimum capacity of the least capacity. So, out of all the cuts that you can potentially create in this network, what is the cut of least capacity? So, if you want to think in terms of this particular example; see for example, this particular this cut here, this cut here this has, what is the capacity of this cut? From 10 to from 1 to 4 is 10 plus 1 to 3 is 20 plus 1 to 2 is 30.

So, the total, the capacity of this particular cut, these are the edges that are getting cut; the capacity of this cut is 60, right. Similarly, what is the capacity of this cut? From 4 to 5 is 5, from 2 to 5 is 5 again, from 3 to 5 is 10 right; 5 plus 5 plus 10 that is 20 is the capacity of this.

Student: Relation matter (Refer Time: 24:22).

Yes, i has to be in S and j has to be in S complement.

Student: (Refer Time: 24:26).

So, the you count the one which is going in, which has i comma j, where i belongs to S and j belongs to S complement.

Student: (Refer Time: 24:36).

Right. So, you say take this cut for example; here I can I have to count 2 the edge going from 2 to 3, I have to count the edge going from 2 to 5 ok and then I have to count the edge going from 4 to 3, because 4 belongs to S and 3 belongs to S complement in this case ok right, that is how I am defining the cut, right ok.

So, I mean you should if you want to think of a present day application; suppose 1 is a node where you know say you have the outbreak of like a corona virus and you want to isolate a destination node from this; these are all the possible ways by which passengers can travel from 1 to from s to t, from 1 to 5.

You want to inconvenience as few passengers as possible you want to cut off routes. So, that as few as little capacity is reduced as possible, I want to inconvenience as few passengers as possible; but yet be able to separate s from t. So, that no more flow can form one can reach, from s can reach t, ok. So, that this is a pertinent application of the min cut term. So, what link should I cut, what is the minimum capacity that I should cut, so that s is separated from t, right?

So, now this becomes, this is the problem of min cut. Now, you can see this problem is also again trying to find at its hard, trying to get to the bottleneck of the network. What we what should you be cutting? Well you should be cutting in some sense; you can see this is also trying to refer to the what is the bottleneck. What is the minimum capacity links that I should cut, so that then the network becomes disconnected; means that you cannot send any flow from s to t, right.

And what is amazing is actually, the min cut problem; the min cut problem you can also write as a LP and it becomes exactly this, this particular one. So, this here, so, the problem here is the Max flow LP; if you write out its dual, this is its dual and the dual turns out to be the Min cut LP, ok.

So, I will leave that as an exercise for you in your homework to verify why this is actually giving you the Min cut, ok. I will give you a few hints here; basically this which vertices lie in S and which lie in S complement have to get determined by the u's. So, if all the vertices that are in S, will have say u i equal to 1 and the ones that are in S complement ok will have u i, let me just quickly write a hint here.

So, you can take u i equal to 1 for i belongs to S and 0 for S complement. And then w ij will be 1 if i belongs to S and j belongs to S complement and 0 otherwise; in which case the objective will compute exactly the cut, the capacity of that cut, ok. And then you what you need to argue is that is in fact optimal; that this is in fact giving you the minimum value of the value of the minima is the capacity of the minimum cut, ok.

So, this is another application of the max of linear programming duality and this famous theorem of called the Max flow Min cut theorem, ok.