Optimization from Fundamentals

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Lecture - 13A
Farkas' lemma

Ok welcome everyone. So, in the previous lecture, we had studied the linear programming duality theorem and the theorem, let me just state it for your convenience here.

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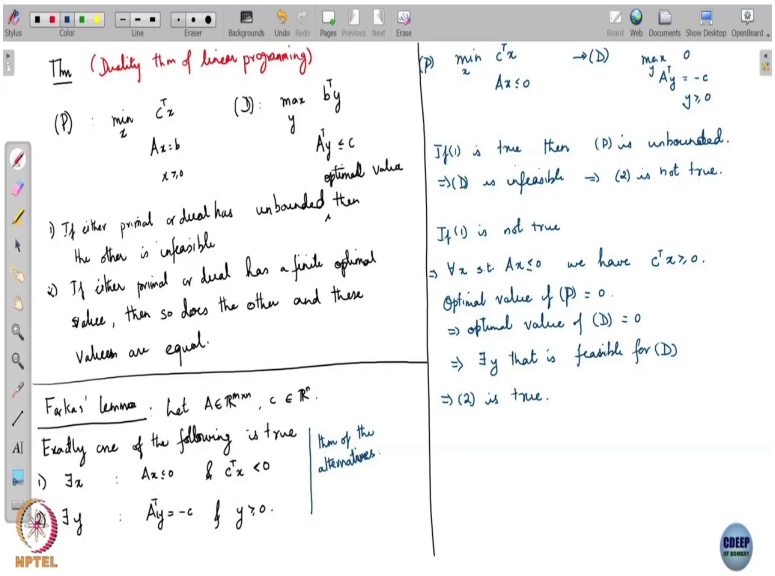

We called this problem P or the primal problem which was minimize c transpose x subject to A x equals b and x greater than equal to 0. This was the primal problem and corresponding to it, we had also what is called the dual problem. So, maximize here the decision variable was x.

Here the decision variable is y, maximize b transpose y subject to A transpose y less than equal to c. And what was the theorem? The theorem said that, there were two parts to it; if either primal or dual is unbounded, then the other is infeasible, this was something we argued very easily from weak duality. And the second, if either primal or dual has a finite optimum optimal solution the value, then so does the other. And these values are equal, right.

This is this was the mean the statement of the linear programming duality theorem; we just write the name here, duality theorem of linear programming.

Student: (Refer Time: 2:48).

Yeah. So, optima when I say it is unbounded, it means that the optimal value is unbounded, is unbound or let me write it as unbounded optimal value, ok alright. So, now this theorem is amazingly powerful actually, which not at all evident when you look at it why this should be the case; but you will see that this is actually at the heart of a lot of optimization. So, I will give you an example of one, it also leads to a number of unusual results.

So, this is an example of an unusual result, it is called Farkas' lemma. Farkas' lemma says the following; exactly one of the following is true. So, it talks of two statements and it says that exactly one of them can hold; first statement is. So, the these both of these statements pertain to a matrix A and vector b ok; so a vector sorry, and a vector c, ok. So, let A be in R m cross n and let c be in R n and exactly one of them is true.

First statement is you can solve this system; so that means there exists an x, such that A x is less than equal to 0 and c transpose x is strictly less than 0. Two, the other statement is that there exists a y, such that A transpose y equals minus c and y is greater than equal to 0; that is the statement of Farkas' lemma.

So, either you can solve this equation, this system of equations has a solution; the first one which says that, which asks for an x x is a n length vector, such that A x is less than equal to

0 and c transpose x is strictly less than 0 or you can have A transpose y equals minus c and y is greater than equal to 0.

You cannot have. So, none of the other cases are possible, which means it is not possible that both are true, it is not possible that neither is true; the only case possible is that exactly one of these is true, ok. So, now this remarkably is actually a consequence of linear programming duality, ok. So, how does this come about from linear programming duality, let me show you this.

So, you consider say this optimization, suppose I look at the problem minimize c transpose x subject to A x less than equal to 0, ok. Now, I have not, we looked at the, we looked at the standard form in as; we looked at the primal problem having a standard form and being a minimization problem, but this problem is more, in this problem has the form of your dual problem actually here. So, just believe me for a moment that, the dual of this one.

So, if this is my primal, then the dual of this one is actually max 0 subject to A transpose y equals minus c and y greater than equal to 0, ok. So, if I if you took this as, you can always write this problem as a minimization problem; put it in this in the form of this primal and then take its dual, you will get back this problem ok, this is another either way. So, just believe me that for that, this is; if this is my primal, then this is it is dual, ok.

Now, this now, what happens if one is true? What happens if this statement one here, this statement is true? If you have, if there exists an x for which A x is less than equal to 0 and c transpose x is strictly negative; what would that mean? Now, the feasible region here A x less than equal to 0 is actually a cone right; because if I can always scale x and it will continue to remain feasible.

So, feasible region is a cone; if I am getting, if I have one x for which c transpose x is negative, then what is going to be the minimum value of this, what is going to be the optimal value of this optimization? Minus infinity, because I can always scale and bring it down to minus infinity, right.

So, it means that the primal is unbounded. If the primal is unbounded, the dual has to be infeasible and dual has to be infeasible means that, there cannot be a y that satisfies the dual constraint; there cannot be a y that satisfies the dual constraints, which means in effect. So, which means in effect that 2 cannot be satisfied, alright. So, what it? So, let me just write it here. So, if 1 is true, then P is unbounded; which means D is infeasible, which means 2 is not true, ok.

So, if 1 is true, then 2; 2 cannot be true, right. Now, let us assume 1 is not true, ok. If 1 is not true, if 1 is not true; which means, what does that mean? What is the negation of 1? It would mean that for all x such that A x is less than equal to 0, the negation of 1; if 1 is not true means, for all x such that A x is greater than equal to sorry, less than equal to 0, we have c transpose x greater than equal to 0.

So, for all x such that A x is less than equal to 0, c transpose x is greater than equal to 0; that would be the negation of 1, right. If 1 is not true, this is what it would mean, right ok. Then what does this say? If 1 is not true, then we have for all x such that A x is less than equal to 0, c transpose x is greater than equal to 0; what does that imply? Look at; let us look at the optimization problems again.

What does it say about P? The optimal value of P will be 0 right; because c transpose x is always greater than equal to 0 on the feasible region and at 0, x at x equal to 0, its value is 0, right.

So, which means, so this would mean optimal value of P is equal to 0. What does that say? Well it says that, the primal has a finite optimal value; which means the dual must also have an optimal value and an optimal solution, right. Which means that, the dual; which means the dual constraints must be satisfiable, right.

So, the dual must have a finite optimal value; which means that the dual constraints must be satisfiable. So, this means, optimal value of P is equal to 0; which means optimal value of D is equal to 0, right. Which means, there must exist; in particular if there is an optimal solution, it means exactly it must be at very least feasible. If there exists a y that is feasible for, it is feasible for D; which means 2 is true. So, if 1 is not true, we got that 2 is true, right.

So, essentially this covers all the cases, you can verify this, if I check one possible case was 1 is true, then it meant 2 is not true; but if 1 is not true, then it turned out 2 has to be true in. So, in no way can you have that both are not true, nor can you have that the case where both are true. So, this sort of result is what is called a theorem of the alternative, of the alternative.

So, it essentially says that out of a given set of alternatives, only one can work, theorem of the alternatives. And it is it became very popular after due to Farkas' himself; because of this very peculiar looking results.

This is just one form of them you, as you play around with linear programming duality; you can derive your own versions of Farkas' lemma of this kind, you know this which says that either this is true or that is true.