Optimization from Fundamentals
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Lecture - 12A
Weak and Strong duality

Today, I will be talking about what is probably the most beautiful and most preformed part of optimization, which is called Duality.

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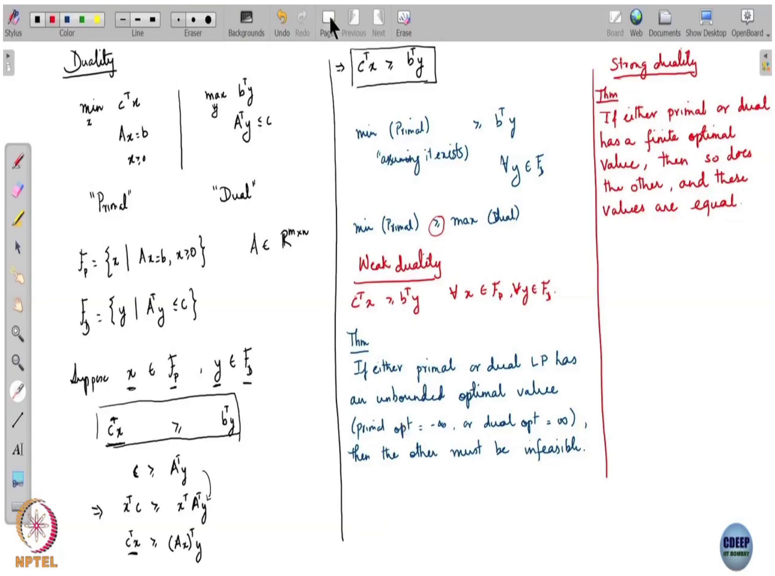

This is a subject that arises specifically in optimization due to the very wave by which we go about deriving solutions of optimization problems. I will explain what this what duality means. So, suppose we have a linear program like this ok. So, today, I am going to limit to linear programming duality.

So, suppose you have a linear program which is written in standard form like this, the decision variable here is x. Now, corresponding to this linear program, I will write another linear program which is this one. So, the first linear program is minimizing c transpose x subject to A x equal to b and x greater than equal to 0, this one is in standard form.

The second linear program is maximizing b transpose y subject to a by A transpose y less than equal to c; subject to A transpose y less than c. This is not in standard form, but it has been written in I have written this particular problem for a specific reason and I will explain what that reason is soon.

So, you have a LP here on the left hand side, which is a minimization LP. The LP on the right hand side is a maximization LP, Now, what we the constants involved here that means, the b that is there in this the objective of this is the same as the b that is in the right hand side here the this we are talking of the same b.

This the c that is in the objective here is the same as the c that is in the right hand side here. The A matrix here is the same as the A matrix here; only thing it has been transposed and written in the constraint. The so, these problems are involve the same set of constants, but arranged in a certain specific way.

You will also notice that here there is a equality constraint because this was in a standard form; whereas, this has an inequality constraint alright. Here, the x is greater than equal to 0; but here, the y is unconstrained, there is no constraint on the side. So, the LP on the right hand side, this is specifically crafted ok. It is very specifically crafted, way with to have this kind of structure and we will soon see what the connection is with the this with the one on the left.

The one on the left, we will call it the Primal LP and one on the right would be called the Dual. Now, here is a quick observation you can make suppose, I gave you any point that is feasible for the primal. So, let us write out some notation here. Let us write this F P as the feasible region of the primal.

So, it is x such that A x equals b and x is greater than equal to 0 and let us write F D as the feasible region of the dual y such that A transpose y is less than equal to c ok. Now, can someone tell me, what are the spaces that these sets line? So, these are in what is the what is the dimension of x and what is the dimension of y?

So, suppose my matrix A is an m cross n matrix, then what is the dimension of x and what is the dimension of y? Yeah, so dimension of x is an n length vector and y is a m length vector right. So, now so these are not these two optimization problems; the primal and the dual, they are not on in the same space at all. I mean one is in R n, the other is in R m; one has n decision variables, the other has m decision variables alright.

Yet, what is amazing is that they are very closely related. So, I will show you this. So, look at. Let us make the first observation here. So, suppose I take a x; suppose, x belongs to this set F P and y belongs to the set F D ok. Now, look at the value c transpose x. Look if I look if I consider c transpose x, now this is the inner product between c and x; x itself is greater than equal to 0 right and x satisfies A x equals b is that right ok. x is greater than equal to 0 and x satisfies A x equals b.

On the other hand, let us look at this value, b transpose y; b transpose y is the is a value attained by a feasible solution y of the LP on the right. So, y just satisfies A transpose y less than equal to c alright, ok. Now, I know that if I know that c look at if I form since y y belongs to F D ok. I know that c is greater than equal to A transpose y for this particular y right. So, I know that c is actually greater than equal to A transpose y.

Now, what I can do is this is a full vector right; c is a vector and y A transpose y is also a vector, every component of that vector if I multiply by A, a number that is non negative, the inequality in the for that particular component will be preserved right. And then, I can add up all those inequalities and get a scalar inequality using all of those.

So, what I am doing? What I am going to do now is just take an inner product with x. x, I know is a vector that is non-negative. x lies here; lies in F P ok. x is a is lies in F P, it is a vector that is non-negative. So, what I can do is I can take an inner product with x. So, which means that will give me x transpose c is greater than equal to x transpose A transpose y right. I am referring to this specific x and this specific y that I have chosen an x in F P and a y in F D.

So, I will just multiplied x on both sides and the inequality. So, took inner product with x and the inequality is preserved right, because every component of x is greater than equal to 0 right so I. So, from here, from this equation, I was able to go to this equation. No problem.

But now, what look at the right hand side, right hand side is actually the same as A x the whole transpose y and let me write the left hand side also better. Let us write the left hand side as since it is just an inner product, let me write it as c transpose x. So, I have basically c transpose x greater than equal to A x the whole transpose y. Now, I make my other observation. Well, again my x belongs to F P; x belongs to F P which means A x is actually equal to b. A x is actually equal to b, since x belongs to F P.

So, what this means is c transpose x is greater than equal to b transpose y. Now, notice what has happened here. You have you started off with any point x in F P and for that, what I was able to show is I took this value c transpose x and I was able to show that c transpose x ok, c transpose x is actually greater than equal to b transpose y.

I started with any x in F P and any y in F D and I got this inequality, that c transpose x is always greater than equal to b transpose y. It does not matter what my choice is. Every x that is feasible for the primal and every y that is feasible for the dual must satisfy that c transpose x is greater than equal to b transpose y.

Now, this let us take this one step further. Observe that the optimal the optimization problem P, the primal optimization problem, primal optimization problem is actually looking for to minimize the looking for the minimum value of c transpose x right. So, if the optimal value exists; that means, it is not minus infinity right.

If the optimal value exists, then the minimum value of the optimization problem is also going to be greater than equal to b transpose y for every y right, so the minimum value of the primal assuming this exists, assuming it exists is greater than equal to b transpose y and this is true again for all y and F D.

But then, what is what let us compare that with the dual problem. The dual problem is looking to maximize b transpose y. Now, since this is true for every y in F D, it is also true for the y that gives you the maximum possible value of the dual right. So, therefore, the minimum value in the primal is greater than equal to the maximum value of the dual.

In short, the optimal values of these two linear programs are related in this fundamental way that the mean the optimal value of the primal cannot go below the optimal value of the dual. The primal is looking to get the least possible value of a certain function of x, the dual is looking to get the maximum possible value of a certain function of y.

But they are they have this kind of tension between them. You cannot bring the primal below the optimal value of the dual and you cannot raise the dual, any higher than the optimal value of the primal ok. Now, this property is what is called we write this in red, it is what is called Weak duality. Weak duality, simply refers to this that for any that c transpose x is greater than equal to b transpose y for all x in F P and for all y in F D ok.

Now, the remember the thing that I want that I mentioned at the start, this these are you might be first tempted to think that you know there is some this is actually somehow is some sort of simple operation. In the sense that say for example, you might be tempted to think like when you are doing minimums minimizing c transpose x, it is like trying to come to the bottom of a function and maximizing see b transpose y, you are trying to get to the maximum of it.

It is as if the same function has been flipped, it is not a flip of this ok. It is not like you are taking a reflection of the objective of one to get to the other. These are two problems written on two separate spaces. So, you are not flipping one to get the other.

There the there is a very there is a specific way in which these they have been crafted, which gets you to weak duality alright. But at the same time, it is not it is really beautiful that you can actually say something like this because it tells you that if there is that you cannot pull down the value of an optimization problem beyond a certain level. If it is a minimization problem; likewise, you cannot raise the value of the optimization problem, if it is a maximization problem beyond a certain level right.

In fact, it gives you this the following results straight away. Let me just state this. If either primal or dual LP is has an unbounded optimal solution has an unbounded optimal value; if either primal or dual has an unbounded optimal value means that the value since we are looking to minimize the primal; that means, primal value is minus primal optimal is value is minus infinity or since and we are looking to maximize the dual; that means, or dual optimal value is plus infinity right.

So, either of if either of these two is true, if either the primal is unbounded or the dual is unbounded, then the other must be infeasible. So, what this means? What this says is that if the optimal value of the primal is minus infinity, then you cannot have even it cannot be that the dual has a feasible solution.

Why is that? Well, what the reason for that is just by weak duality. If the primal value is minus infinity; that means, the minimum here is minus infinity, then you would get that minus infinity is greater than equal to the maximum value of the dual. Now, in particular, it would be greater than equal to the value of the dual for any y right, for any y in F D.

So, but a finite value of y would give you a finite value of b transpose y. There is no way that can be less than equal to minus infinity. So, the contradiction is that there cannot be a single y that satisfies the constraints of the dual which means the dual has to be infeasible. So, if the primal is unbounded, then the dual must be infeasible.

Likewise, if the dual is unbounded, so dual takes value plus infinity, then there cannot be a finite value of x which satisfies the constraints of the primal because then, you would get that you have plus infinity here on the right hand side and something here which is finite, but greater than equal to the plus infinity right. So, that is also impossible.

So, in short, if any of them is unbounded, so primal taking value minus infinity or dual taking value plus infinity, then the other must be infeasible right. So, this also gives you a way of testing if your problem is going to have minus infinity is going to have an unbounded optimal value. Because it amounts to checking if a bunch of other linear inequalities or equations can be satisfied right.

So, if just giving you the just as soon as I give you this LP here, the primal LP, I can I need to I can check if this is going to have a minus infinity as the solution. Well, one if this if A transpose y is less than a less than equal to c is satisfied by some y, then it means that the dual is not infeasible. Then, there is no way that this one can have minus infinity as a solution right; likewise, for the dual alright.

So, this theorem the is what is called the first property is what is called the property of weak duality and this here the theorem that I just wrote is just simply a consequence of weak duality. Now, but the theory of duality does not end here because there is an even more powerful result that is out there and that result actually says that these two these two problems, if they admit solutions, then their optimal values are actually equal.

Weak duality simply says that the optimal value of the primal is greater than equal to the optimal value of the dual. It just only guarantees you a an inequality here. But a stronger property is true which is which simply which says that if both are finite then they are both equal alright. So, that is what that property is what is called the property of Strong duality; strong duality.

And the strong duality theorem is this; if either primal or dual has a finite optimal value, then so does the other and these values are equal. So, the if the primal has a finite optimal value, then the dual must also have an finite optimal value and the two values should be the same.

Likewise, if the dual has a finite optimal value, then so does the primal and its optimal value is the same as that of the dual. So, if one and from the earlier theorem, we know that if it does not have a finite optimal value, means it has if it is unbounded, then the other one must be infeasible alright ok.

So, now what we will do today is prove the strong duality theorem ok. So, that is the agenda for today. The strong, I will prove the strong duality theorem and I will also give you some an application of the strong duality theorem ok. Now, the one of the strong reality theorem actually rests on one key property of convex sets and the property is that way very is very easy to see; but it is not that easy to prove.

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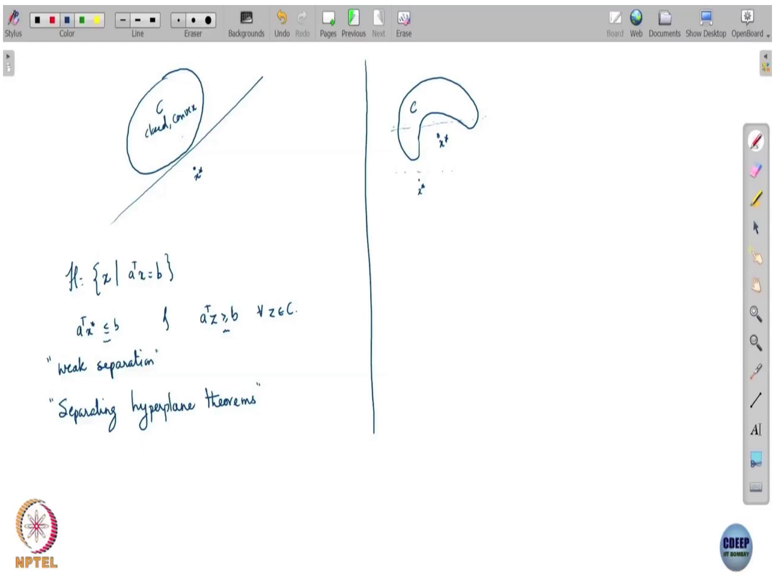

So, suppose, I have a convex set here c ok. Suppose, c is let us say c is closed and closed and convex suppose, I take a point here x that lies outside c, a simple observation is that I can always draw I can always find a hyper plane like this, a hyper plane that separates the point x from c. What do I mean by separates? Separates means that the point x should lie on one side of the hyper plane and the set c in entirety should lie on the other side.

So, my hyper plane suppose is; so, let me call this point something else. Let us call this x star. My hyper plane is suppose x such that a transpose x equals b, then if then this by separation of the set from of the set c from x star, what we mean is well a transpose x stars is less than equal to b, say it lies on below the hyper plane and a transpose z is greater than suppose greater than equal to b for all z in c; the entire set c lies on the other side.

Now, the way I have written this is this form of separation is where I have allowed both these inequalities to be weak, which means both these both the point it is possible for the point x star to lie on the hyper plane, it is possible for the set c to also touch the hyper plane touch. But not cross the hyper plane right it can.

So, you can have equality in both. This is called weak separation. This property is what is called weak separation; but you can also talk of much stronger versions of separation and I will we will use a stronger version of separation. In the stronger version, both these inequalities will become strict you will.

So, it will be like the kind of picture, I have drawn here. Your point x star lies on the other in one half space, but not on the hyper plane and the set c also lies in the other half space and now does not touch the hyper plane ok. This is what is called these theorems that guarantee you something like this are what are called separating hyper plane theorems; separating, they are called separating hyper plane theorems and are one of the key tools for proving bounds in optimization ok.

So, or optimization and several other fields also, where if you want to show that something is impossible, try to show you one of the key ways of useful tools in that is through separating hyper planes. Now, what is the role of convexity in a separating hyper plane theorem? You can see that by just draw looking for looking at another case, suppose this set c was not convex like this; but say suppose shaped like this and my I had a point x star of the out here.

Now, is it possible for me to separate x star from the set c by a hyper plane? No, it is just not possible. No matter how much you try; you try to, you will never be able to find a hyper plane that separates the two. You can find some other curve that separates the two; but not a hyper plane right. This is the beauty of convexity. Convexity guarantees you this ok

Now, so, we now there are other points that may be separable. Like for example, a point x star that lies here this may it may be possible to separate from the set c ok. But it is not possible for every point outside the set right. So, you know when you have when your set is convex, you can say it can be separated from every point that lies closed and convex it like, it can be separated from every point that lies in the exterior.