Optimization from Fundamentals

Prof. Ankur Kulkarni
Department of Systems and Control Engineering
Indian Institute of Technology, Bombay

Lecture – 1C
Review of real analysis (sequences and convergence)

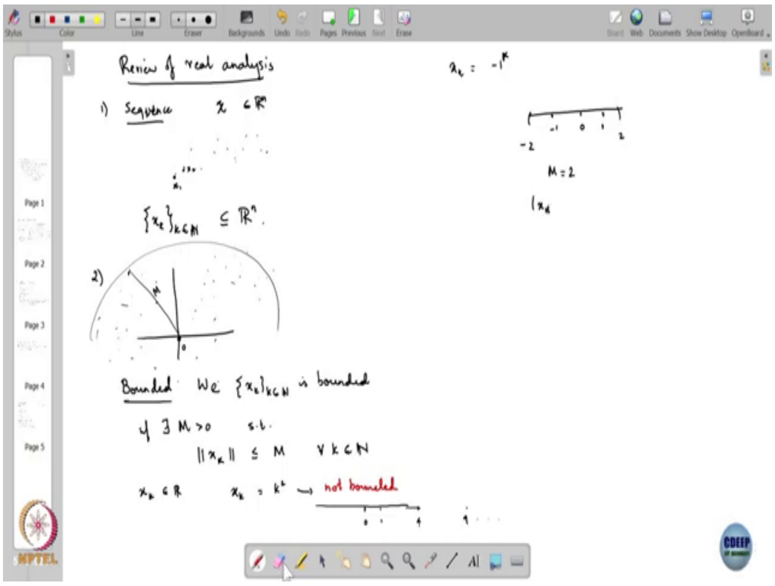
So, what we will do is see as we go into the go into the subject, we will need a little we will need keep needing concepts from some or the other branches of mathematics. So, I will introduce some of these concepts in the beginning ok. We will also tell you what it means for a solution to exist and when is there is solution and so on.

So, you have to, so thankfully there are some blanket theorems that we can apply which guarantee the existence of a solution. So, you have to make sure that your problem the problem you have at hand satisfies the assumptions of those theorems, and then you have existence occurring, no problem.

You do not have to, so this is the existence question can become a chicken and egg problem because you are the existence should not require you to claiming the existence should not require you to actually compute what you are hoping exists right. So, you should have a way of claiming existence that does not require finding the solution in the first place ok.

So, thankfully there are such there are such results. So, we will go through those. So, we will start with this.

(Refer Slide Time: 01:43)



So, first simple idea we need is that of a, so I will start with a just a review of real analysis and later also some convexity ok. So, the first concept we need is that we need is that of a sequence ok.

And in this course, we will stick to two problems that are in Euclidean spaces so in R n a lot of what I will teach you can also be applied in larger spaces ok, but we will try we will mostly stick to stick to R n. So, I will talk of a sequence here which is in R n. So, the way you will write this as so you have a point x that lies in R n.

Now, you have and a sequence is just a series of such points ok. So, this is so x. So, here is a point x 1, there is another point x 2, etcetera etcetera ok, x 3 etcetera. So, a sequence is denoted like this x sub k with and on outside the bracket you write k in n. So, you have so the sequence is an infinite series of points x 1, x 2, x 3, etcetera etcetera, all lying in a common space. So, all of them say suppose belong to R n.

Now, since you are in R n, we can talk of how far is this sequence. There is a sequences have a certain direction in which they go right as k increases you can sort of think of them as a moving particle or whatever or a fly that is flying around in this room.

So, at time 1, at time 2, at time 3 etcetera, where is this particle currently, you can this is how you can track a sequence. And so you are since we are in R n we can, so here is your the origin of your space and we can talk of how far is where is the sequence headed right.

So, as first we say that the sequence is bounded you say it is bounded if you have the following. So, if you look at this particular figure, this sequence is traveling all over the place, the indices keep going on and on till infinity, the sequence keeps traveling. We say it is bounded if there is some big ball centered at the origin, if there is a ball centered at the origin in which it in which the entire sequence resides ok.

We say x n is bounded if there exists an m let us say greater than 0 such that if I take the that is norm of x k, norm of x k is nothing but the distance of x k from the origin. Norm of x k is less than equal to M for all k in natural. So, we say it is bounded if it is if the sequence if the norm of x of every point in the sequence is no more than M, there is a. So, important notice the order in which these are this is being stated.

We want a common M that will work for all k. We want there exists an M such that for all k norm of x k is less than equal to M alright. You can of course, find an M that is tailored to a to each point, so then that is that does not make the sequence itself bounded. That just means that every point has a finite would have a finite norm alright. So, here we are asking for a common M such that the norm of x k is less than equal to M for all k in n.

So, the picture you should have in mind is that there is this ball of radius M here in which this entire sequence is lined. Of course, the ball is not necessarily minimal or in any sense it is just some ball of finite radius ok. It is not minimal in any sense I mean that it does not mean that you cannot find a smaller ball or anything like that all that matters is that the entire sequence is within the ball in that case we say it is bounded.

Now, I said that you can we can follow the sequence along its progression and as k increases, and you can ask where is this sequence headed. So, simple answer is firstly is it at least not running away, eventually is it not running away. For example if you have a sequence x k say for example in R. Let us take a sequence in R.

Say x k equal to k square right such a sequence what would happen to such a sequence? This is a sequence that keeps increasing, increasing as k increases. Eventually if you look at it along the real line, here is your 0 of the real line at k equal to 1 it is at 1; at k equal to 2, it is gone to 4; at k equal to 3, it is gone to 9 and so on so; eventually it will just it will it keeps blowing up increases and increases. There is no M that you can specify such that all the elements of the sequence are within M right within have normally have absolute value less than M ok. So this is this sort of sequence is not bounded.

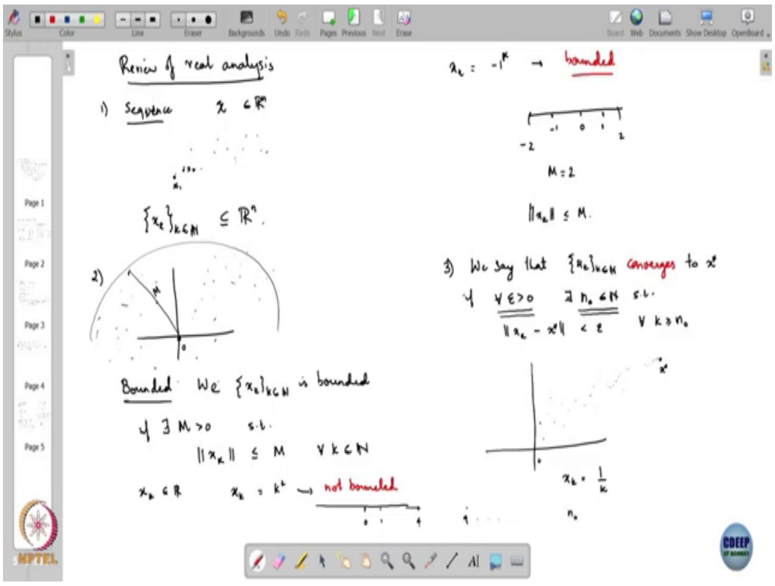
Let us take another example. Consider the sequence x k equal to minus 1 to the k, minus 1 raise to k. How does this sequence behave?

Student: It oscillates

It oscillates between plus 1 and minus 1 right. So, I at k equal to 1 it is at k equal to 1 it is at minus 1; at k equal to 2, it is at 1; k equal to 3, it goes back to minus 1; 4 at 1 etcetera, etcetera. Is this sequence bounded?

Right, it is bounded. It is bounded because you can I can put all these all the elements of that sequence are can all be enclosed within this plus minus 2 region right. So, I can take for example, M equal to 2, then and then the absolute value of x k or the norm which is the absolute value of x k would be less than equal to M. So, this sequence is, this sequence is bounded.

(Refer Slide Time: 09:50)



Student: (Refer Slide Time: 10:08)

Now, just because it is bounded does not mean the sequence is headed somewhere, sequence could still dance around all the time without having any particular direction. But interesting sequences are those were are those and which we are of always interested in or the sequence is that may initially dance around for a long time, but eventually head somewhere right.

What this is saying is, so we say that x k converges to x star if the following is true. You tell me you specify a distance you specify how what it means to be how what it means to be close to x star well you specify it by saying ok, well, I want I wanted to be within the distance between x and x star to be no more than epsilon, you thus you specify the epsilon ok.

So, for every epsilon that you specify, now it is for me a challenge. I have to now come tell you does the sequence is the sequence going to be in within epsilon of x star. Unfortunately, usually it is not the case that the entire sequence is going to be within epsilon of x star. But then maybe not the full sequence, but does the sequence eventually end up within epsilon of x star, within epsilon distance of x star.

What does it mean for it eventually to end up, so if for every epsilon that you throw as a challenge to me, I should be able to produce an n naught, I should be able to produce an n naught such that from the point n naught onwards, the sequence completely lies within the within an epsilon distance of x star.

So, from epsilon from n naught onwards I am within this. So, the picture you should have in mind is, so again let us draw a pair of axis, here is your origin, here is the sequence. Initially it can do whatever it wants maybe eventually it the it has this behavior where it seems to be tending towards some point and that is it. I cannot even draw this more finely.

Here somewhere here is the x star. It is headed, headed, headed, headed here, and then eventually after a point all the elements of the sequence are accumulating around x star. And eventually since for every epsilon, I should be able to find this find an n naught like this. It means that however, small you make the epsilon infinitely many of points of the sequence are actually within epsilon of x star within epsilon distance of x star, only the first n naught are going to be outside right.

And however, small I make the epsilon means I can make the epsilon 10 raise to minus 10 and still within that are going to be infinitely many points and outside only finite many. I can make a 10 raise to minus 20 is still within that are going to be infinitely many and outside are going to be only finitely ok. So, from the set, there will be a point onwards from which it is going to be inside that right.

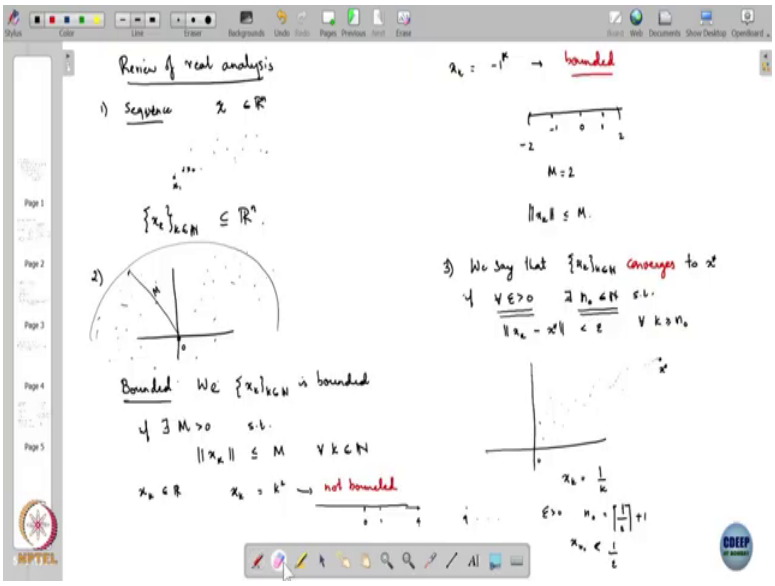
So, let us take another let us take an example. Let us take x k equal to 1 by k ok. Again the sequence of real numbers x k equal to 1 by k. What does this, does this converge and it if it converges, converges to what? Converges to 0; why does it converge to 0? As see as k increases it becomes smaller and smaller that is while that is true, we want to say we need to say it based on this definition, based on the definition that are stated.

So, for every epsilon greater than 0, can you find for me an n naught? Can you find for me an n naught such that for beyond that beyond n naught all the x k s for k greater than n naught have absolute value less than epsilon?

Student: (Refer Time: 14:48)

Yeah, exactly.

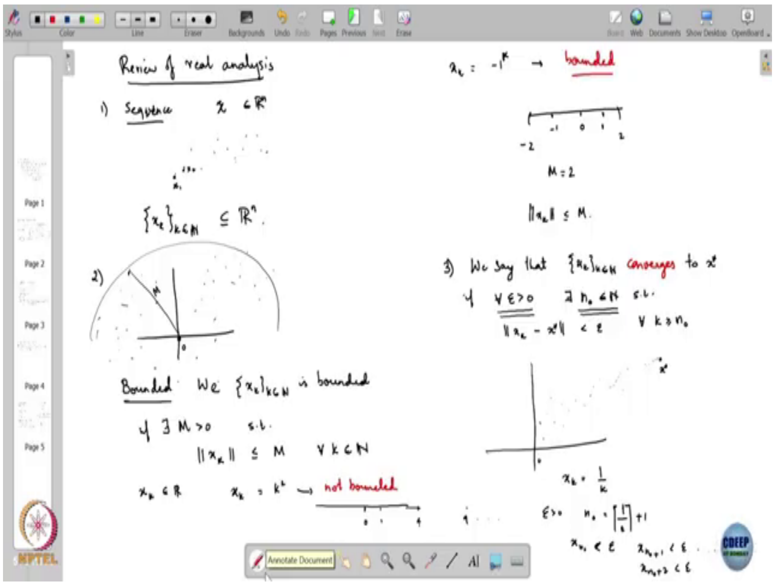
(Refer Slide Time: 15:03)



So far so what I can do is, so what I need is that I need to find an n naught, so you are given giving me an epsilon given me an epsilon which is positive, and I need to find an n naught. What will I take my n naught as what I can do is, I can take n naught as yeah I can do this right. So, I can take n naught as 1 by epsilon, and the seal that to the greatest integer, sorry the smallest integer greater than 1 by epsilon ok.

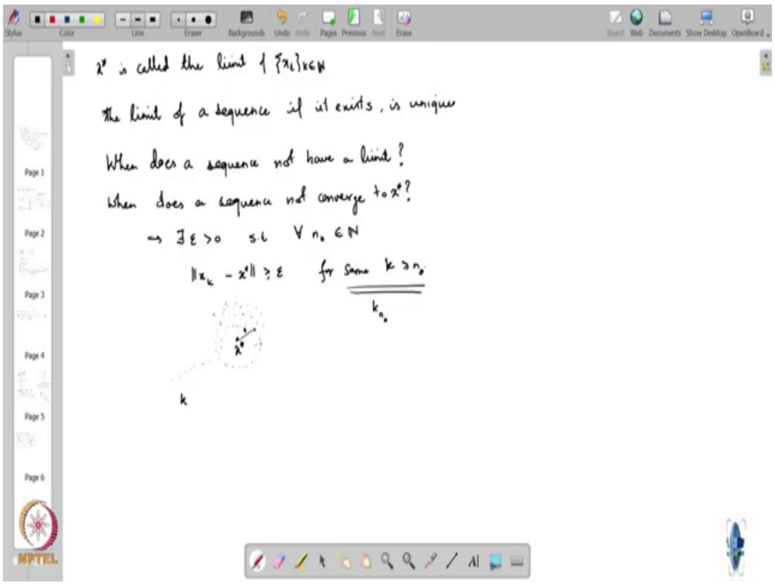
Then what would happen? From, so x n naught itself is going to be definitely less than x n naught is going to be definitely less than equal to 1 by epsilon or I can make this plus 1. So, this is definitely less than 1 by epsilon x n naught is less than 1 by epsilon and not only x n naught x n naught plus 1 x naught plus 2 etcetera all of them are sorry not 1 by epsilon sorry epsilon sorry my mistake.

(Refer Slide Time: 16:23)



x n naught is going to be less than epsilon and not only that x n naught plus 1 x n naught plus 2 x n naught plus 3, all of them are going to be less than 1 by epsilon right. So, in short the sequence as you can see is going to it may have large number of values that are away from 0, but infinitely many from a certain point onwards all of them are going to be as close to 0 as you like right.

(Refer Slide Time: 16:59)



So, x star in this case is called the limit of the sequence. It is called the limit of the sequence x k ok. And it is very easy to show that limit of a sequence if it exists, is unique. You cannot have a sequence that converges to two different points I did not specify this and it is not ah it is generally it is not needed because we are talking of distances between points between x k and x star.

Student: (Refer Time: 18:01).

No, no, no, we have n, but then they have to be if you have point in a different space, then no no you need to you need to embed this all of them in the same space. So a distance like this is meaningfully provided both distance between two points is meaningful provided both have the same number of coordinates at least.

So, both must have n coordinates in this case ok. So, we talked of, so what it means to converge, so limit of a sequence if it exists is unique. Can you tell me what it means for a limit to not exist? When does a sequence not have a limit? That does not say that the limit does not exist, that just says that the limit is not x star.

Student: So what?

So, let us first do the simple thing. What does it mean for a sequence to not converge to a given point x star? It means that there exists. Sorry, so let me write the simpler version. So, the answer to this is there exists.

Student: epsilon

Greater than 0.

Student: Such that for all (Refer Time: 19:45).

Such that for all n naught.

Student: in natural

In natural numbers.

Student: x k minus.

X k minus x star.

Student: is less than.

Greater than equal to epsilon for.

Student: for all k for some k is (Refer Time: 20:06).

For some k greater than n. What does this mean? So, when a sequence is not converging to an, it may converge but it certainly not converging to this particular x star that I am considering ok. When a sequence is not converging to this particular x star, it means that I can find an epsilon ball around that x star and epsilon radius region around that x star such that no matter where you start, no matter which n naught you start from.

There is always going to be at least one k after that which escapes from thatregion. There is going to be one x k after that n naught such that x k escapes from that. Is this clear?

So, what it, what this means is? So, the way this looks is, so here is suppose my x star and I should be able to find some epsilon region around it ok. So, here is this epsilon radius.

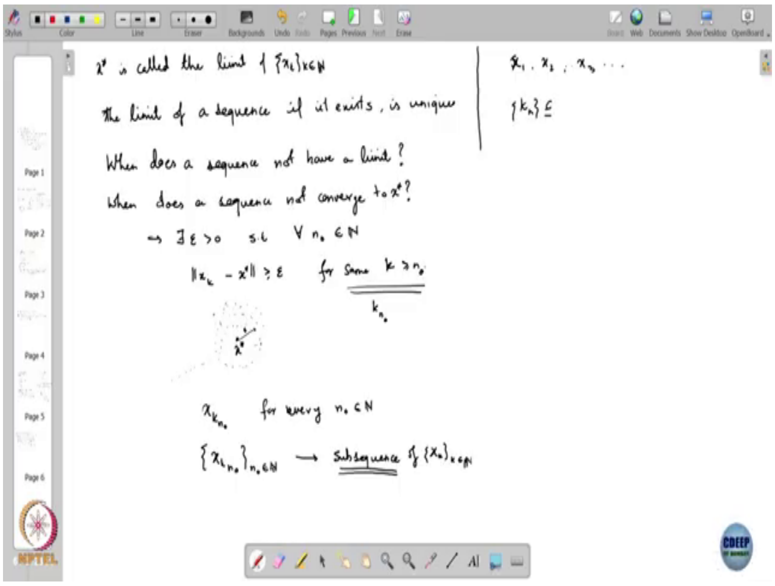
So, and the way the sequence will behave is that sequence may come inside, but goes outside; goes outside once, maybe come inside again, goes outside again, come inside again, but goes outside again etcetera, etcetera. Point is it never eventually lies inside. So, it keeps going out infinitely many times.

So, you look at the sequence from any n naught onwards. You will always find one point from there onwards which is escaped from any n naught onwards, no matter how larger n naught is. So, you can never find an n naught such that the sequence from here onwards is forever inside ok. So, there is such a sized ball for which this hold ok. And usually this such a so to make this true the ball has to become would have to become very, very small.

So, this will become easier and easier if the ball is smaller because it will becomes easier for the sequences points to escape from it right. If you throw your net very wide, obviously many points will lie inside, but you there exists such a; such an epsilon that is the that is very that becomes easier to show once your epsilon becomes smaller ok. So, the sequence keep as keeps escaping like this ok.

Now, if you look at this, this these points that have escaped ok, so for every n naught I can find one x k with k greater than n naught that has escaped right. So, let me just call that something. So, let me call that; let me call that this particular k this k let us just index it by let us call it k n naught, k sub n naught ok.

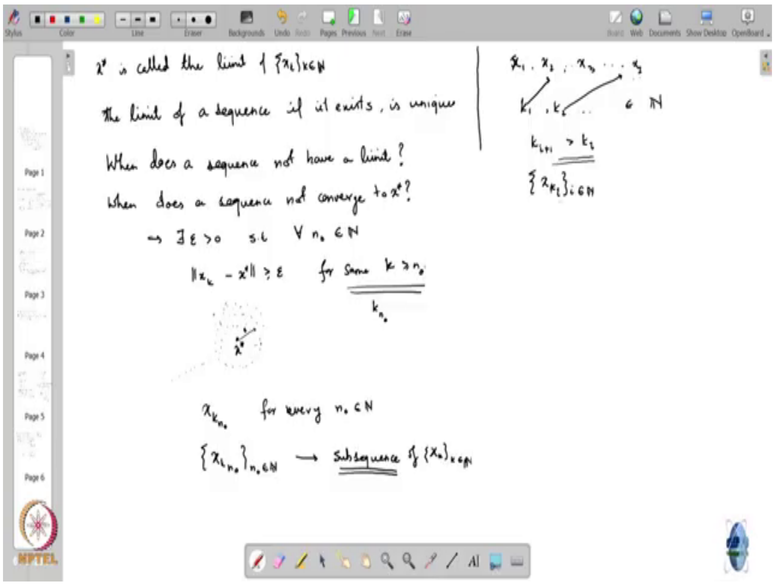
(Refer Slide Time: 23:40)



So, what it means is this there is always an x k n naught which is gone out right, and not only the there is such a x k n naught for everyone n naught right, for every n naught. So, this itself right these points this set of points itself can be thought of as a sequence right, so they are traveling this is a, so this here is a an example of a sub sequence of x k of the original sequence.

So, a sub sequence, a sub sequence is formed like this. You what you have your you have your sequence x 1, x 2, etcetera, x 3, etcetera, the way you construct a sub sequence is by construct you take a subset of the indices. So, you take k n you consider a sequence of natural numbers like this.

(Refer Slide Time: 25:10)



You consider the k 1, k 2 dot, dot, dot, these are natural numbers such that this k i plus 1 is strictly greater than k i right. So, when you index the subscript here, you the natural number jumps by a positive amount k i plus 1 is strictly greater than. Then you can with this once I have this. So, this is now a strictly increasing sequence of natural number, I can use these as indices on my original sequence and that gives me a subsequence. See if I.

Student: (Refer Time: 26:01)

So this is the just a subtle technical point. If I did not have this point here where I am asking for strictly greater right the sequence could go backwards also. So, that changes the direction of moment motion of the sequence. So, we do not allow that in the definition of the subsequence.

If you have equality, the sequence talls. Although i is getting index, k i remains the same, so essentially you are taking repeating the same point from the original sequence and we do not want to allow that. We want the subsequence to move alongside the sequence and in the same direction.

So, k 1, k 2, k 3 are just indices; these are natural numbers. These are just some natural numbers. So, I am going to use these two index k to index my original sequence. I am going to look at the sequence at k 1, k 2, k 3 etcetera. For example, k 1 could be 2, k 3, k 2 could be 7 suppose, etcetera. So, as I pull this as I put look at the sequence with these indices only, what I get is a sub sequence of the entire sequence.

So, we will stop here now. We will continue this in the next class.