Optimization from Fundamentals
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Lecture - 10A
Extreme points and optimal solution of an LP

So last time we were talking about linear programs and I showed you that I said I defined a linear program as simply this.

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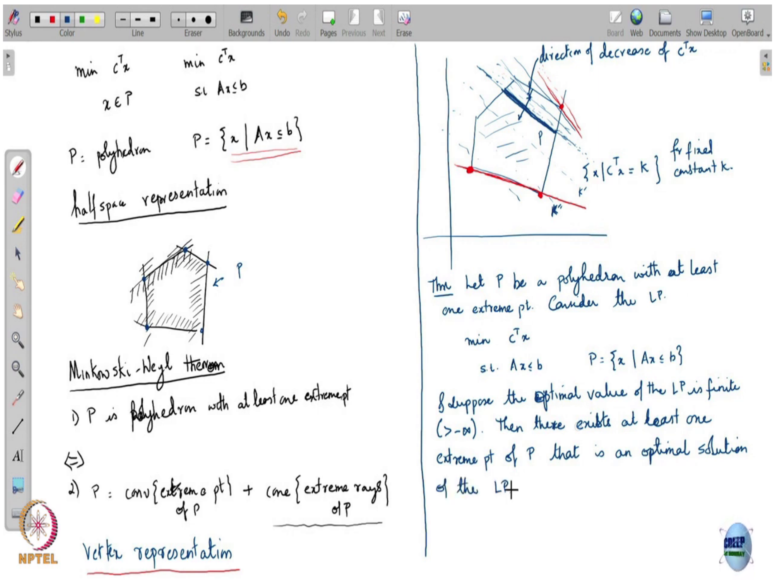

Minimizing a linear function over a set P which was a polyhedron P is a polyhedron. And this sort of problem takes the form in general takes the form minimize c transpose x subject to A x less than equal to b. So, the constraints that define the polyhedron are these. So, the set P is x such that A x is less than equal to b.

Now, this is what is called the half space representation of a polyhedron half space representation. What is that mean? So, if I give you a poly the way I am defining for you a polyhedron is through the half spaces that, whose intersection if I take gives me the polyhedron.

So, this is for a these are for example, this is my the poly I have drawn the polyhedron and these are the half spaces that defined right. So, I am defining for you the half spaces, whose intersection if you take they gives you back the polyhedron.

Now, the thing that I want to show you today is that actually there is another way of representing a polyhedron. Which is the another way of representing the polyhedron which is which came which we saw that actually through what is called the Minkowski Weyl theorem right. What is the Minkowski Meyl theorems say Minkowski Weyl theorem said that P is a polyhedron.

This is equivalent to the claim that saying that P is a polyhedron with at least one extreme point. That is equivalent to saying that P is the convex hull of its extreme points, plus the cone generated by its extreme rays, extreme points of P and the extreme rays of P. Now, if I if the polyhedron is bounded then this term here which we call price of the cone generated by the extreme rays this term is just a 0 and then P is simply the convex hull of its extreme point ok.

So, if a polyhedron if you have a bounded polyhedron; that means, if you have a polytope and a polytope is simply can be described as the convex hull of its extreme point. So, this set here the way is actually a poly is actually a polytope and it is every point it in it can be written as the convex combination of these 5 points these extreme point right.

So, this now this way of representing a polyhedron or a polytope most specifically this way of representing a polytope as not through the half spaces that define the polytope, but rather through the extreme points these extreme points. But this way of expressing a polytope is what is called the vertex representation ok.

Now what I will show you now is that, if you had the vertex representation of a poly of a polytope ok. The problem linear programming would be very easy you would be it would be very easy to know what the optimal value is and at and come up with at least one solution for it for your problem ok. Now, if you the so what that would mean is that the challenge in linear programming is that we do not is in coming up with the vertex representation right.

So, you are given usually a problem that is in the half space representation, you know the polyhedron is defined using the half spaces that sub that form the faces of the polyhedron ok that is the and what we do not know is where the vertices lie, they have to be computationally discovered as part of the solution method this is clear.

So, the all the so solution method is effectively trying to find or enumerate the vertices of the polyhedron the extreme points of the polyhedron that is ok. So, let me just show you why this is the case that if you had the work space representation or if you more generally if you had a representation of if you had a description of all the extreme points and the extreme rays of the polyhedron, then linear programming would be trivial to solve ok.

So, let us just look at this ok. So, this basically so in order to talk about this let us first. Let us just get a intuitive picture first. So, I will just draw a neater diagram of the same thing here. So, I have suppose a polyhedron like this; this is my polyhedron. Now I am optimizing a linear function over it and now let me plot. So, this is my polyhedron it is in some space say let us for simplicity here it is in r 2. I what I am going to plot is the contours of the linear function that I am going to optimize right.

So, I am going to plot contours that look better of the form that they. So, this contour here this is some c transpose x equal to k x is such that c transpose x is equal to k for a fix constant k right. So, what this means take any point on this contour and the value of the objective on that for that point is k right. Now you can draw this contour for these kind of contours for various values of k.

So, you have this is k dash this is k double dash etcetera etcetera. Now what do you what can you observe about a contour like this. So, let us keep our focus on this one, let us look at this contour what can you observe about it.

So, it intersects the feasible region; that means, the poly the polyhedron over which you are optimizing it intersects the feasible region here right. These are in this is the. So, these are all the points that are in the polyhedron and give you objective value k. Now question is are any of these optimal?

Can any of these be optimal? The one can any of these give you the minimum value of k? See if you think about it a little bit you will realize that it is not this cannot be optimal, I do not need to tell you what the value of k is for that. Whatever be the value of k these cannot be optimal the reason is because this is a linear function.

It should be that the function either increases in this direction or in this direction right. So, nearby here either to the north of the function either above this contour or below the contour, you should be able to find a value that is either greater than k or less than k right.

So, you should be able to get values k plus minus delta for an you know small enough delta in the by perturbing this contour a little bit. And by perturbing the contour a little bit you are not going to lose feasibility, you are going to still be inside the you are still going to have points in P right.

So, what this means is a have contour like this it cannot comprise of optimal solutions. So, suppose for example, the since I am talking of decrease minimizing the function let us suppose that the function is decreasing in this direction ok. So, this is the direction of suppose this is the direction of decrease of the function c transpose.

So, which means that if I as if I look at the contour here or a contour here etcetera as I go in this direction I will keep getting lower and lower values of k ok. So, what this means is in order to optimize a function I need to search in this direction right I. So, I need to look for a smaller contour with the lower value of k and a further lower value of k.

And then finally, it should happen that I eventually get to a point like this what sort of point is this. This is a point where the contour just passes only through my extreme point right, the contour just pass passes through only my extreme point. Now, if I look for a value of K that is even lower even lower than this then I will not I would not it would not the contour will not even intersect the feasible region right. So, then you are that is a value that is not even feasible.

So, below this if I look for a contour that has value less than this; this is not going to intersect the feasible region this is clear. So, what is this observation tell you this observation tell you that there must be a solution of this linear program of minimizing c transpose x over this polyhedron right.

If you look at that linear program where you minimizing c transpose x over this polyhedron there must be a solution of this linear program on an extreme point of P right. There must be at least one extreme point of P that is a solution of this linear program.

Now, it can happen or that there can be multiple for example, you know it can happen that the contour is shaped in such a way that you know when it actually pass. Say for example, let me draw it here the contour has a is the slope is such that it when it passes through this extreme point it actually coincides perfectly with this phase with this edge here right. And then therefore, it will end up passing through this extreme point as well.

So, in that case both of them will be optimal solution. In fact, all of these will be optimal solution that can also happen, but there will at least the one which is an extreme point and an optimal solution right.

The main lesson here is that the thing that we expect from a linear program is that, let us say consider the case where you have only optimizing over polytopes; that means, your over bounded polyhedron that. The kind of the nature of solution that we expect is that there is going to be a extreme point that is a solution right.

So, now come back to this way of representing this thing that I mention which is representing a polyhedron using its vertices or using its extreme points alone. So, if someone gave you not these constraints like this, not the constraints Ax less than equal to b, but rather a vertex representation. That means, a list of; a list of points that have the extreme points of your polyhedron then optimizing the a linear function over that would be trivial.

You would have to just check go through the list enumerate the values of c transpose x for each of those extreme point output the list right, that would have; that would have optimize the function right. So, the challenge in linear programming comes down to precisely this issue, that we do not know where the extreme points are and they have you know do not know means that they know them implicitly.

We know that they are extreme points of this sort of set, but the exact formula for them or the exact values of them have to be discovered by doing some calculations and some on top of this these constraints right given these numbers right. So, they have to be discovered as part of the from the problem data .This is clear? Any questions about it?

So, now let me answer let me address the more general case suppose you have not a polytope, but a poly you know a general polyhedron which is which could also be unbounded and ok. So, let me address that and so, let us try to let us just write out a theorem here. So, let P be a polyhedron with at least one extreme point. Consider the linear program minimizing minimize c transpose x.

So, the Ax less than equal to b and suppose the optimal value of the L P is finite. We are talking of an L P where you are minimizing the objective. So, the optimal value of this L P is finite means that effectively it is not minus infinity. So, it is definitely greater than minus infinity that is what it that is what it means right. So, you can you cannot get keep getting lower and lower as low value as you like that sort of thing is not possible. So, that is the sort of case we are considering

So, let P be polyhedron with at least one extreme point and we are considering an L P whose value is finite this is clear ok. Then there exists at least one extreme point. So, here I should write P is there exists at least one extreme point of P that is an optimal solution of the L P.