Optimization from Fundamentals
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Lecture - 9B
Minkowski-Weyl Theorem

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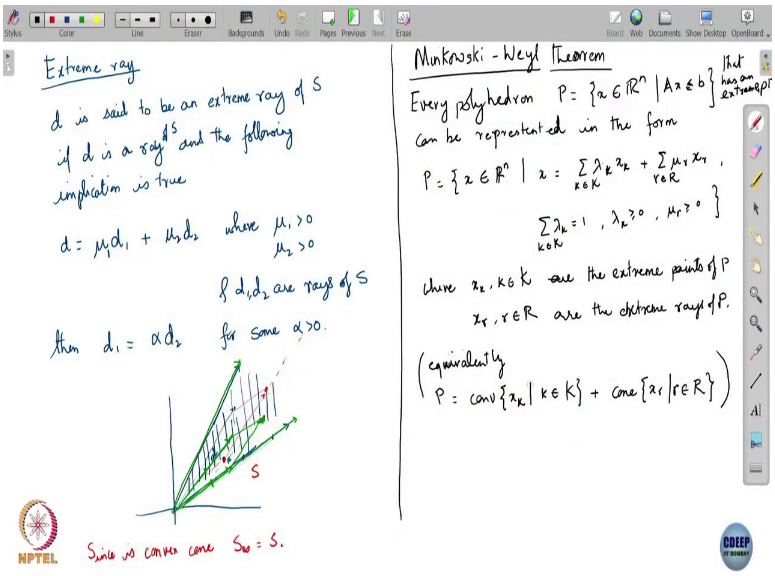

An extreme; an extreme ray so, d is said to be an extreme ray. So, d is said to be an extreme ray if d is a ray firstly, to begin with and the following implication is true. What is the implication?

So, if you can write d in the following form. So, write d as mu d 1 plus mu 1 d 1 plus mu 2 d 2 where mu 1 greater than 0, mu; mu 2 greater than 0 and d 1 d 2 are miss this here d is said to be an extreme ray of the set S ok. If d is a ray of S and the following implication is true d is.

So, if you can write d as mu 1 d 1 plus mu 2 d 2 where mu 1 is positive, mu 2 is positive and d 1 d 2 are rays of S, then it has to be; it has to be what? What should be true? Then d 1 should be equal to some alpha d 2 for some alpha positive ok.

So, when we are talking of rays, an extreme rays and so on, it the problem becomes trivial as soon as when one has a bounded set because all these will become 0 on their own ok. So, the these are all discussed only in the context of unbounded convection ok.

So, when we are talking of so, and so, consequently 0 is really not considered a ray unless you know it is for pathological reasons. So, here, we are talking of therefore, d 1, d 2 etcetera all non-zero vectors. So, a ray d, this vector d is said to be an extreme ray of S if you so, if you can if the following implication is true that means, you write it as a conic combination.

A strict conic combination of two other rays of the set, then it has to be that the two rays are actually coincident means one is just a scaled version of the other. So, these are vectors yeah. So, ray is after all is just a vector.

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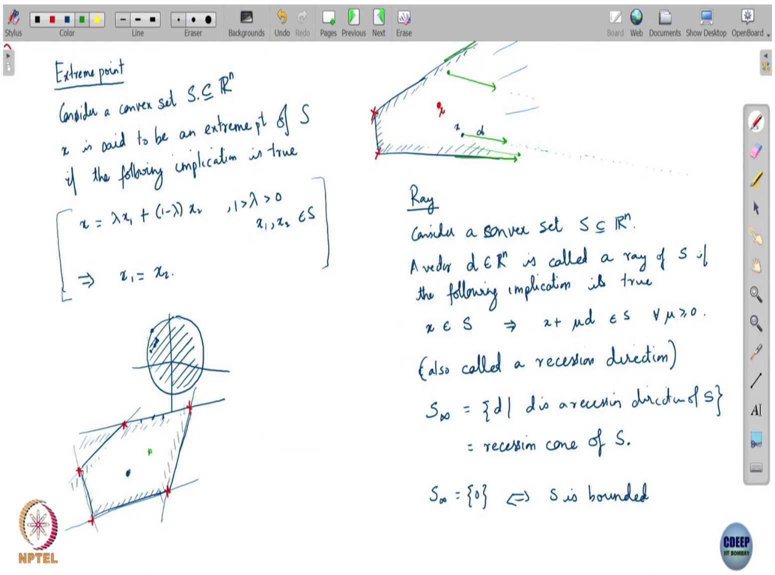

So, go back to the definition. A vector d is called a ray if the following implication is true that is it. So, when you are adding the rays, you just add them as vectors yeah. So, if you can write, if you happen to be able to write d as a conic combination of two; two rays, then it has to be that that that the two rays are coincident in that case, we say that the that the ray d is an extreme ray of the set S.

Now, in this example, I will need to be, I will need to show you take a bit of time to determine what the extreme rays are. So, let me, but let me show you another example.

So, consider so, if you just for simplicity, let us just take a cone, take a cone like this, a convex cone like this. What are the extreme rays of this set? Yeah, so, if you take; if you take these two any of these vectors, you know take this vector, take this vector all of these are extreme rays right.

So, generally one does not talk of a when we are talking of an extreme or an extreme ray or a ray of a set, really what we are only referring to is the direction so, it does not matter. Because you can scale a ray and will it will remain a ray so, all you need to really care about is the direction. So, this direction you can say is an extreme ray right. So, all points on this direction are the extreme ray.

Why is that the case? No, the cone comprises of all of these point ok. To let me ask you a simpler question first, tell me all the rays of this set before we talk of extreme rays? What are the rays of this set? Yes, so, what is the; what are the rays of set?

Student: (Refer Time: 05:57).

So, the set of all rays of the set is the set itself right. All you can because this is a convex cone, you add, they this is closed this is a convex cone, you can take conic combinations of any two points in it, it will remain in the set it is ok. So, you can take any point, add to it, any a scalar multiple of any other point in the set that is just going to be a conic combination, that conic combination will be in this way ok. So, this for this cone, the set of all rays is actually the set itself is this clear ok.

So, now, what is this? So, I will just illustrate this for you, here is the point x for example, here is my vector d, if I have to add d; if I add d to x, I am effectively going somewhere in the to just complete this parallelepiped here, this is where my point ends up and I can keep doing this, I can take any scalar multiple of d and I will still remain in this way right.

So, effectively, I am just translating x along the direction d forever and I will remain in this way and this will be true for any x that I take and any d that you take within the set ok. So, the so, in this case, this is this set S, since S is a cone, is a convex cone, S infinity is actually S, set of all research recession direction is like 0 set itself ok.

Now, what are the extreme rays then? Extreme rays are those that if you try to write them as a strictly positive conic combination of two other rays. Then then those it cannot be done those two rays have to be; those two rays have to be coincident on each other right.

So, which means that if you take a say if you take a say take some point like this, this sort of point cannot be an extreme ray, this sort of point cannot be an extreme ray because I can always think of this as the as a vertex of some parallelepiped like this. This is a conic combination of this one plus this one or linear combination of these two, so some conic combination of these two for example.

So, if I take a so, any of a point like this here, cannot be an extreme ray. So, if you start thinking about it, you will realize that the only extreme rays possible are actually the ones these are extreme rays and these are extreme rays ok. The word is well chosen, they are sort of certain extreme, you can imagine them they sort they are the ones that flying in the cone right.

Now, this is in r 2. Now, things in r t; r 3 and higher dimensions become a lot more complicated. So, for example, if I have an ice-cream cone right, this is an object in r 3, what would be the extreme rays of this cone? You can see for example, all the ribs of the cone, you know all the directions that start from the apex of the cone and go along the shell of it, all of those are extreme rays of; extreme rays of the cone ok.

So, if I likewise, suppose if I constructed a cone in r 3 which in which I had say which was not an ice-cream cone, but rather you say a hexagonal cone with six sides to it, what would be the extreme rays of such a cone?

Student: Hexagon (Refer Time: 10:30).

The edges that you would get you know once again the ribs of the cone, you know the edges that you would get from that define the hexagon, those will be the extreme rays. Anything along the surface like this which on the flat surface is not going to be an extreme; is not going to be an extreme ray because it could be you can write it always as a combination of two other extreme rays right, two other rays.

All the rays of the set S can be written as a conic combination of all other rays the that the answer to that is yes. But I will explain in a much in a better way because this becomes a, this sort of statement is it anticipates what I want to just say and it is very sharp, can be made of very sharp for polyhedrons. So, let me come to come to that point now ok.

So, what I have, we were building up basically to what is called the Minkowski-Weyl theorem, this theorem is due to Minkowski and Weyl. So, in the previous class, as I was leaving, I asked you to imagine what a bounded polyhedron would look like right. A polyhedron is an; is an intersection of half spaces. What is a bounded polyhedron? How does that look? Not a polygon.

So, a bounded polyhedron would look something like this right, it would look like a set like this ok, this is for example, not an bounded polyhedron and what do I mean by it looks like something like this? What special about this set? You take any point in the set right, you can write it as a convex combination of these red corner points or red extreme points right.

So, every, once you have; once you have a bounded polyhedron, you can just look at its extreme points. And you can generate the entire polyhedron by taking convex combinations of those extreme points ok. What if you have a set like this which is not a bounded polyhedron, but an unbounded polyhedron?

Now, it is not true anymore that this set has just two extreme point. Now, it is not true that every point can be written as a convex combination of these two red points right. There are points here for example, which cannot be written as convex combination of these two.

So, what is the analogous extension of that earlier observation? The observation is that if you have a bounded polyhedron, then its every point can be written as a convex combination of these fixed points, you know these extreme point plus yes. So, you have a so, see so, what were you saying? Very nice, very good.

So, the thing that he is anticipated is actually the statement of the theorem. See, if you have a point like if you have a polyhedron like this which is unbounded and take a point like this here ok, take this point x, this point can be cannot be written as convex combination of just the extreme points.

But it can be is what he saying that it can be written as convex combination of the extreme points plus translation with the rays that you have in the set. And in fact, you can take only the, it is enough to take only the extreme rays of the set ok. So, this is basically the main observation and that this theorem is that the that this theorem is talking about.

So, I will just state for state the theorem for you. Every polyhedron P ok, so suppose you polyhedron, I am going to represent it in in the following form, I am going to represent it as x in R n such that Ax is less than equal to b can be represented in the form sorry my mistake here.

So, I am, I need to tell you one blanket assumption that under which this holds, and I have not stated here. So, every polyhedron so, we are talking of polyhedral that have extreme points that has an extreme point ok.

So, every polyhedron that has an extreme point can be represented in the following; in the following form. So, the polyhedron is actually the in the following form, every point in it is a convex combination of points x k where k ranges over a set capital K and plus a conic combination of points x r where r ranges over a set capital R ok.

Now, here the points x k where k belongs to capital K, these are; these are the extreme points of this of the polyhedron and x r where r belongs to R are the extreme rays ok. So, if you so, what does mean?

This you can you just take the convex hull formed by the extreme points of the set, look at also the cone generated by the extreme rays of the set and add the two together, that actually gives you back your polyhedron ok. So, this is what the this theorem said. This is true provided the set has an has at least one extreme point ok.

So, you take a so, equivalent way of writing this equivalently P is equal to the convex hull of the points x 1 till sorry points x k where k belongs to; convex hull of point x k where k belongs to capital K plus the cone generated by points x r where r belongs to capital R.

So, every polyhedron has this sort can be represented in this kind of form. So, this is an extremely important theorem because it tell gives us in one short of view a way of viewing all polyhedron. Every polyhedron is basically a polytope which is a bounded polyhedron created from just the extreme point, translated along the cone created by its extreme rays right.

So, you all so, to describe a polyhedron completely, all you need are the base polytope that is formed by its extreme points and the extreme rays that are there in the polyhedron, put the and translate one, translate the base along the directions defined by the cone. And you are done that is how you define describe a polyhedron ok. This is an extremely complicated, powerful theorem where it also seems very intuitive, but it is actually very hard to prove ok, it takes quite a few pages to actually do the proof ok.

So, the way as I said the you go back to this; this sort of this kind of set here, this sort of set the way this what Minkowski-Weyl theorem is saying is that the way this set is generated is that you take the convex hull of the extreme points which is just this segment and keep translating this segment along the two extreme directions or the two extreme rays. What are the extreme rays? In this case this and these are the extreme rays.

So, you take this segment, translate it along this direction, translate it along this direction that will basically generate for you or translate it along the any direction that is; that is formed by linear convex conic combination of these two directions, take this direction, this direction, take conic combinations of those translated also along those, all of that put together will give you the entire; entire polyhedron.

This yeah, I am just I mean one would have to calculate this, but yeah, to me it looks like this ray and this ray would be extreme rays.

Student: (Refer Time: 23:22).

Which one?

Student: (Refer Time: 23:26).

If you take a ray below it, then it will not be an a ray of the of the set.

Student: (Refer Time: 23:33).

Yeah, the one inside is not, anything inside here is not going to be an extreme ray.

Student: Yeah, you know what I am saying this ray which is like horizontal ray which is the bottom one which is represented as (Refer Time: 23:45) combination of two rays inside; inside the.

Which ray are you referring to the one here this one?

Student: Let us draw a straight line inside (Refer Time: 23:52).

Ok. I let us, I can do it, but this one?

Student: Yeah, (Refer Time: 24:03).

This if it is inside the, if it is in the interior that is not. So, the exact the extreme rays are these the ones that I am calling extreme rays are these direction ok. So, that gives you. So, that is the basically, that is Minkowski-Weyl theorem. Now, why is this so important? Because it actually in one short gives us also a way of understanding what happens in the first kind of optimization problem with inequality constraints.