Optimization from Fundamentals
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Lecture - 9A
Polyhedrons

So, we ended our previous lecture by defining what is called a Polyhedron, I will just revise that for you quickly.

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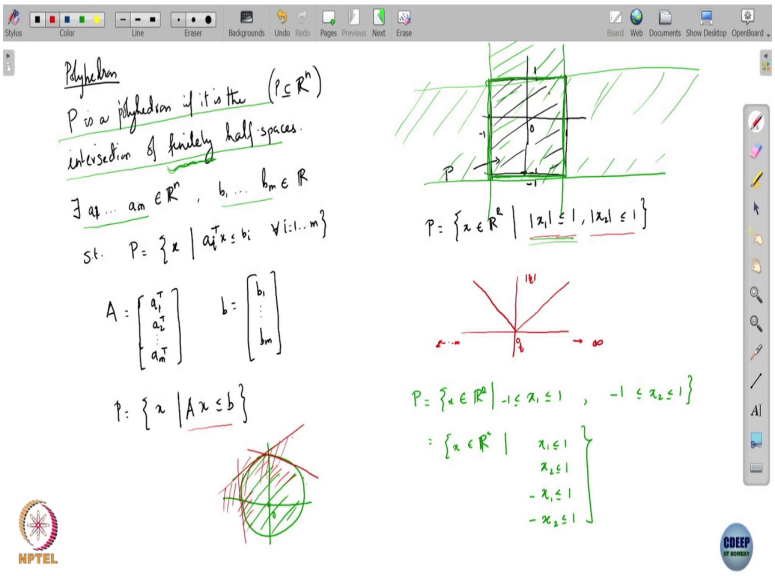

So, a polyhedron P is a polyhedron is if it is the intersection of finitely many half spaces alright. So, if you can find finitely many half spaces; so that; such that their intersection defines is equal to P, then that then P is a polyhedron. So, what this means is there exists some say m of these; m of m vectors; a 1 till a m in R n.

Let us assume P is a subset of R n; there exist these m vectors in R n and also m scalars like this b 1 till b m such that P can be expressed as the common region that belongs to all of these; these half spaces, defined by these vectors and this these constants. So, P is x such that a i transpose x is less than equal to b i; for all i going from 1 till m.

This in we; as I said we can consolidate these a's by writing them in the in this way; you write say for a 1 transpose as the first row of a matrix, a 2 transpose as the second row and so on; all the way down till a m transpose that gives you a matrix A. You put the b's together in a vector called b and then that gives you that P is actually nothing but the region x defined regions; defined by x such that A x is less than equal to b ok.

So, it is the region that is defined by finitely many linear inequalities. Now, this is the this sort of way of defining a polyhedron where we first say it is the intersection of half spaces and then you go about finding the half spaces and give you that gives you this sort of representation.

Now, polyhedron may not have may have multiple representations. So, on the face of it what looks like a poly what looks like it is not a polyhedron could also end up being a poly right. For example, here if you look at this; if you look at this kind of representation, this is saying that the polyhedron is all the x's that satisfy some linear inequalities right.

But then you can also have a set like this. So, let me show you for example, let us look at this set in R 2. So, this is plus 1, this is minus 1, 0, minus 1, plus 1 and I am looking at this way; so minus 1.

So, this region; what is this region? Well, this is one way of representing this region; let us call this set P, this region let us call this P; one way of representing it is to say well this is x in R 2; sorry R 2 such that, if I look at the absolute value of the first coordinate that is less than equal to 1 and the absolute value of the second coordinate that is also less than equal to 1.

So, the absolute value of the first coordinate and the absolute value of the second coordinate are both less than equal to 1; that is actually this region right. Both coordinates have any point in this region; you take a point say here for example, or here wherever you want; it if you look at its coordinates it is in absolute value they are less than equal to 1; that is actually this region. But if you look at these inequalities, the absolute value of x 1 less than equal to 1 and the absolute value of x 2 less than equal to 1.

These are not linear inequalities right; they can they are not of this kind because the absolute value function is not a linear function, the absolute value function is a the absolute value function. How does the absolute value; what does the absolute value function look like? It looks like this right.

So, this is absolute value of t and here is t; this is 0, this is going to plus infinity, this is minus infinity right. So, the absolute value function is not a linear function. So, if you look at these inequalities; I have written P as an as the common region of two non-linear inequalities right.

So, it so but that does not mean P is not a polyhedron ok; the point is P is still the intersection of half of finitely many half spaces and what are those half spaces? Those half spaces can be seen here itself. So, you can take this as this as one of the half spaces, you can take this as another half space; take this as another half space and then finally, this as another half space. The common region in all of them is actually our set P right; no a transpose x less than equal to b.

The definition of a half space is that it is a transpose x less than equal to b right. So, you so what we have is that this P is; if you look at it, in one way it is defined using as the common region in; as the common region of non-linear inequalities; whereas, if you look at it in another way; it is actually the intersection of finitely many half spaces. This is a; so because it is an intersection of finitely many half spaces, it is a polyhedron ok; that is.

So, the point is this; this here is our definition of a polyhedron and from here we all we can see is that there exists some way of representing it; as a as an intersection of finitely many half space ok; that means, there exists a 1, a 2 up till a m and b 1, b 2 will b m such that their intersection gives you b ok.

Now, they may not be the one that you have chosen at the moment; here you have chosen some inequalities, they are non-linear inequalities, that does not mean there are no linear inequalities that will describe the same region right. So, in particular you can see observe here that P is also the intersection of these inequalities right.

So, x; the absolute value of x 1 is less than equal to 1, if and only if x 1 is less than equal to 1 and also greater than equal to minus; sorry less than equal to 1 and greater than equal to minus 1. And likewise for x 2 also; so these are actually the half spaces that I have just highlighted here.

So, these are; these this is this here is one of your inequalities, this is another inequality, this is another inequality and this is another inequality. So, there are total of four of them; you can put them together in the systematic form; x in R 2, x 1 less than equal to 1, x 2 less than equal to 1, x 1 minus x 1; I will less than equal to 1 and minus x 2 less than equal to 1 right.

So, those define our four half spaces; this is clear ok. So, a polyhedron is the intersection of finitely many half spaces; now this as I and of course, you objects this; what this example showed you was that objects that on the face of it may not be written as an intersection of linear inequalities can still be turn out to be polyhedron ok.

Now, another thing to point out here is the importance of finite. If I do not allow; if I allow for infinitely many half spaces, then many many sets end up being polyhedron which are not really what we want them to be ok. So, for example, just take the circle; there is circle with this, the region inside. I take a tangent here, I can this circle is actually the intersection of all of these regions right.

This region defined by this tangent; this region defined by this tangent etcetera; this region defined by this tangent etcetera, etcetera. I draw a tangent at every point; I will get infinitely many such tangents, infinitely many such half spaces. Their common region is actually the circle; the circle is not a polyhedron because it is not an intersection of, it cannot be written as the intersection of finitely many half spaces ok.

So, now this is one kind of issue related to representation where you have polyhedron that I can you can have a set which is represented by non-linear inequalities and can still be a polyhedron. The other type of issue of representation which I is what is extremely important for optimization which is what I want to talk about right now. So, this is the for this; we need a couple of concepts.

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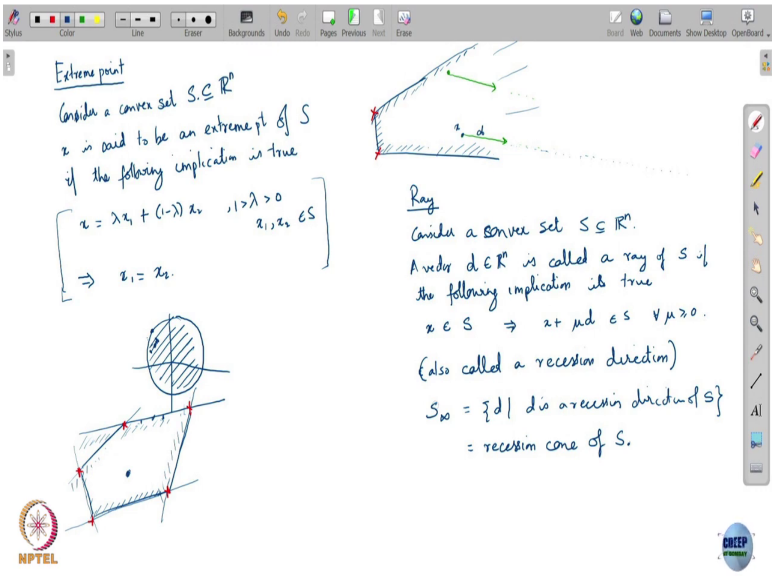

So, let me introduce this concept for you first is what is called an extreme point. So, consider a some convex set ok; now x is said to be an extreme point of S; if the following is true; following implication is true; that means the implication is that x can be written as a linear combination of a convex combination to be precise.

So, a of two other points in S; so x is equal to lambda x 1 plus 1 minus lambda x 2; where lambda is strictly greater than 0 and x 1 not equal to x 2 belong to S. So, if x can be written in this way; then it should imply ok, sorry if x let me remove this; x can be written as in this way where x 1, x 2 belong to S, then this should imply that x 1 equals x 2.

So, if x can be written as the convex combination of two points using a strictly positive weight for each of them. So, x can be written as some lambda which is x 1 plus 1 minus lambda x 2, where lambda is strictly positive ok, then it has to be that x 1 equals x 2 and when x 1 is equal to x 2; they are actually both equal to x itself right. So, an extreme point is a point that cannot be written as a strict convex combination of two distinct points in the cell ok.

So, an extreme point is one for which this implication holds. So, whenever you try to write it as a convex combination like this of two different points; you well, you are it has to be that those two points are actually the same and it is equal to equal to the point itself as a convex; as a strict convex combination of two points, then the two points have to actually coincide ok. So, what does an extreme point look like; let us take for example, just I was just writing drawing a circle; takes the circle as a convex set, what are the extreme points of this set?

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Every point on the boundary is an extreme one right because the circle does not have; if you take any point on this on the boundary if you try to; if you were able to write it as a convex combination of two other points, then the circle would end up having would end up having a flat surfaces.

So, this sort of point cannot be written as a convex combination of two other points in the set. Every point in the interior can of course, be written as a convex combination of two other points; just always there are always two other points around it use you know whose midpoint will be this point.

So, every point on the boundary of the circle is actually an extreme point of the circle. So, the extreme points actually are more interesting when we are talking of polyhedron. So, if I draw a polyhedron like this is an intersection of half spaces ok. So, the half space is being this, this half space, this half space, this half space, this half space and likewise this half space.

So, now tell me what are the extreme points of this polyhedron? Once again a point here in the interior cannot be an extreme point. Oh yes of course, sorry I said convex combination, but did not make that clear; yeah yes lambda is strictly between 0 and 1; yes.

So, any come back to the polyhedron here. So, this sort of point is not an extreme point. What about; what are the other points which cannot be extreme points. Yeah, so if I take some point here in the middle of an edge here; this sort of point is also not an extreme point because I can write it as a convex combination of two other points like this.

So, eventually if you think about it what remain as extreme points are actually only these points; this one, this one, this one, this one and this one right; these are the extreme points of this polyhedron, let me draw for you another kind of polyhedron. So, this is an unbounded polyhedron right; so what are the extreme points of the set? Let us say this goes on forever; what are the extreme points of this?

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It has only two extreme points which is this one and this one; these are the extreme points of this polyhedron right ok. Now, this actually; this figure is actually very interesting, if you think about it that why this polyhedron is unbounded. So, what does that mean?

There is a sort of a direction here along which if you keep going you know; this sort of direction. Just imagine, you started off from here and you keep kept proceeding along this direction. You keep going; you will remain in the set, there are directions like this for a polyhedron ok.

Once it is unbounded, the very fact that its unbounded means gives you that there will be some direction along which you can travel right; such that you will never leave the set right. So, and what is amazing is that these directions are such that they do not; it does not matter where you started from; means so, if there is a direction like this; let us say a direction like this along which if you can; so if you started from a point here and went in this direction forever; you will remain in the set.

But, I can go along that direction; starting from some other point also in the set and I will still remain all forever in the set. So, for instance I can start from this point here and still go in this direction and remain in the set right. So, there are directions like this once polyhedron is unbounded, there are directions such that if those directions are such that you can start from any point in the set, keep walking along that direction and you will never leave the polyhedron.

These directions are what are called extreme rays of the polyhedron sorry these are these directions are what are called rays of a polyhedron; I will sorry not extremely, they are called rays of a polyhedron or are also called recession directions. So, a ray ok; so this actually this particular thing that I just mentioned, this does not really require the set to be a polyhedron.

So, long as it is a convex set you; if there is a direction along which you can keep going, starting from some point; you can go along that direction starting from any other point and you will still remain in the set. This is the shape of; this is the nature of the shape of convexity ok; so a ray is defined like this.

So, let us see consider a convex set, let us take that convex set here R n. So, consider a convex set or a vector d in R n is called a ray of S; if the following implication is true. So, x belongs to the set S implies that x plus mu times d belongs to the set S for all mu greater than equal to 0; sorry yes.

Student: (Refer Time: 22:44).

Yeah; so if it is unbounded, then if it is bounded, then the only such vector will be d equal to 0. Now, so once x is in S; this x plus mu d is in S, for all mu greater than equal to 0. So, the; so x plus mu d is; so the way you should look at this is the here is your point x, here is the direction d; x plus mu d as mu get is since as you go from for mu greater than larger and larger mu, you keep going along this, this dotted line that I just drew further and further from x along the direction d ok.

So, for all mu greater than equal to 0; this point belongs to this x plus mu d belongs to S ok. So, this such a vector is what is called a ray yeah. So, another name for it is called d is also called recession direction and the set of all recession direction is; of a set S is often denoted by S infinity, this so all d such that d is a recession direction of S.

This actually is turns out to be a cone and it is called a the recession cone of S. It is very easy to check that this is going to be a cone and it is called a recession cone of S yeah. So, if the set is bounded; then the only vector that satisfy that would satisfy this sort of condition is the vector 0 right.

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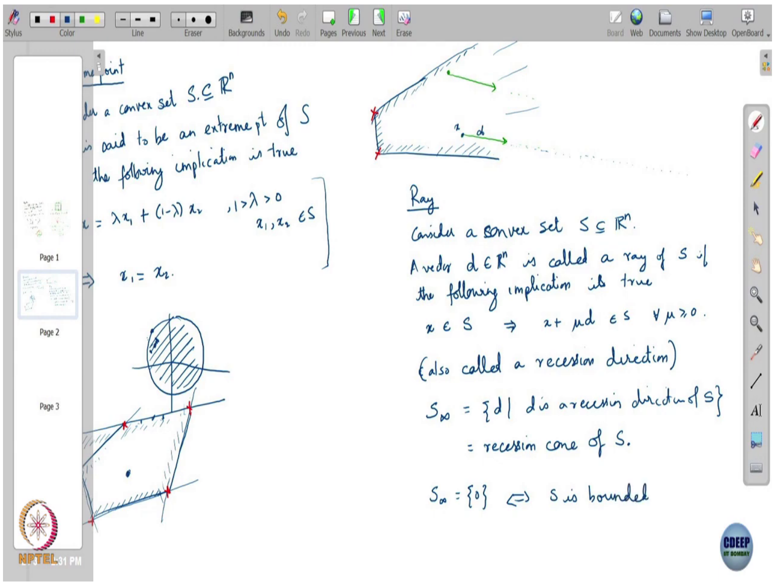

So in fact, it is this is this statement is also true that S infinity; this cone is as is equal to just the 0 vector with the singleton set with only the 0 vector; if and only if S is bounded.