Optimization from Fundamentals
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Lecture – 8C
Convex Analysis - III

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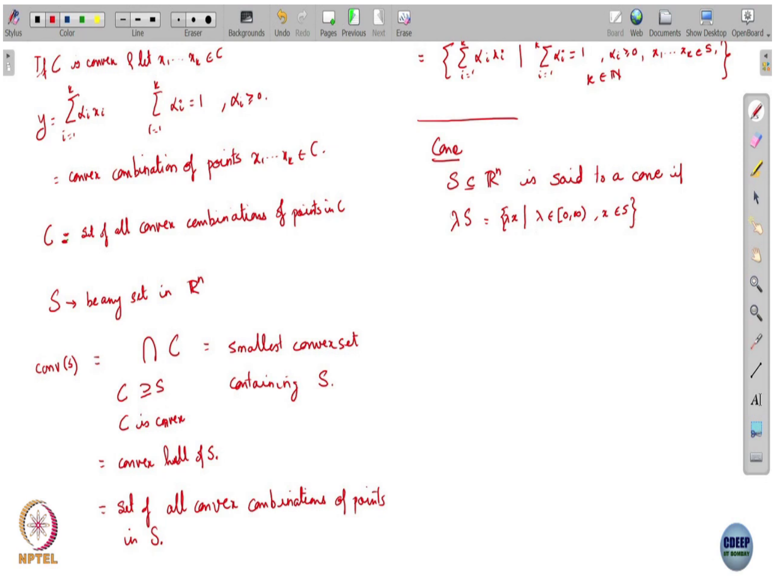

So, let us go to the next. So, this is the definition of a Convex hull of a set. Another concept we need is that of a cone, set S subset which is in R n is said to be a cone if the following holes. So, if I what I should what I want to do is, I want to take some alpha times S or let me use a different notation let me use lambda. So, take lambda times S what this is essentially is I am taking every vector x in the set and scaling it with lambda.

But then I am going to scale it with a non negative lambda. So, lambda times S is just the set S magnified or shrunk by lambda by a factor lambda right. So, every vector x has been this multiplied by a scalar lambda ok.

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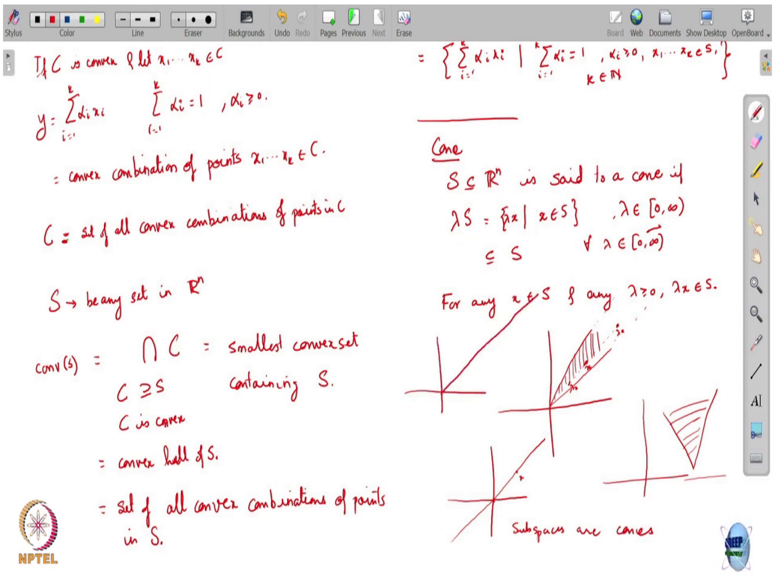

So, my mistake here I do not need the and a, but I am talking of lambdas that are non negative ok. So, lambda S is the set which is where every vector has been multiplied by lambda. Now the set is said to be a cone if lambda S belongs to S or is a subset of S. So, what does this mean? Belongs to S for all S for all. So, what does this mean? So, you take any x another way of saying this is for any x S and any lambda greater than equal to 0.

The point lambda x also lies in S, take any point x in S, take any lambda greater than 0 greater than equal to 0, then the point lambda x must lie in S that is what then the set S is said to be a cone. So, what does the cone look like? So, cone looks like a cone right. So, but it does not have to look like a cone.

So, for example, a set like this going all the way, right. So, if I take any vector x here and if I say suppose I scale this guy this vector it will scale reach here maybe if I scale it by lambda. So, if is this is my x this will be my lambda x or maybe it will scale get scale down if my lambda is less than 1. This could be my lambda x all such points are all in S ok.

So, this sort of set is this is a cone. Any other examples of cones? Yeah, line passing through the origin that is a cone that does not look like a cone right. So, this is the line passing through the origin this is also a cone ok. Why? Because I can take any point x here scale it, it lies on the line yes. So, now, can cone not has pass through the origin. That is not possible because by the definition I am allowed to take lambda equal to 0 here.

If I have to if I can take lambda equal to 0, then it means that you know put lambda equal to 0, then here I am getting the left hand side is just 0. So, 0 must be in the set right. So, cone that is whose apex is lying somewhere other than the origin this is not a cone for us this sort of set ok. So, affine sets for example, are not necessarily cones because they do not have to pass through the origin.

But all subspaces are cones so, but. So, you so subspaces are cones these sort of sets are cones. You can you do not need to take the full subspace though because you are allow you are not multiplying by negative scalars. So, I am not allowing for lambda negative here. So, I am not scaling in the in this direction the opposite direction here right. So, I so what that means, is a set that looks like this, say just a ray like this, this is also cone ok, although this is not clearly not a subspace ok.

Now is a set is a cone convex all these examples were convex, what about is a cone always convex? Can you give me an example of a cone that is not convex? Yeah, I get what you are saying, but probably the word base is not quite the right word see the. So, what he saying is base of a cone need not be convex what you are saying I can roughly relate to what you are trying to say.

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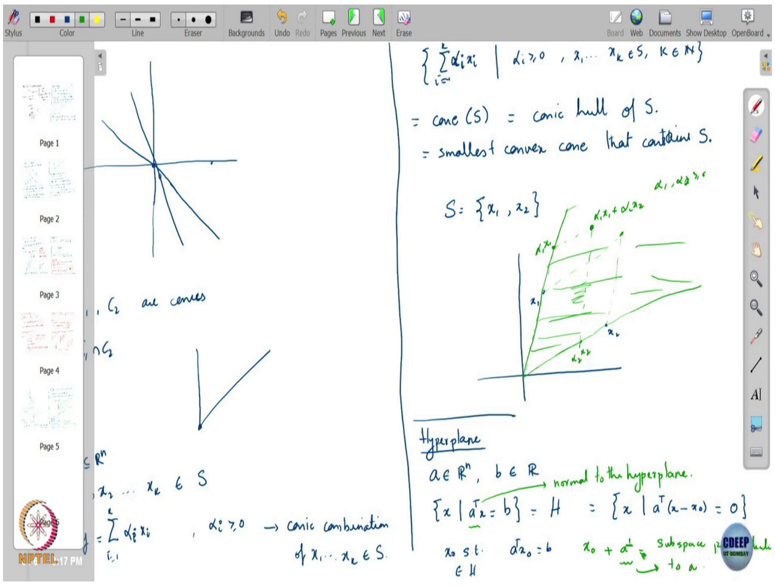

Let me give you an example first in r 2 and then we can come to this more exotic example. So, what I have drawn here these are axis, this itself is a cone right. So, this is my origin I take any take any point scale it by any scalar lambda greater than equal to 0 that gives me a point on the axis right. So, this is a cone I can all not only that I can take some other cross like this passing through the origin these two.

So, the it is union of this line with this line right, that is also a cone right. So, this is also; this is also a cone. So, cone need not be convex, what you are describing is a slightly more is a slightly more interesting sort of object. So, you can it is not easy it is not easy to do it in r 2; in r 2 there are limitations.

So, I can try to explain to you how it would be. So, think of just think of some non convex shape say for example, some star or something. Just a cutout of a star on a page shrink it down all the way down to the origin and also grow it all the way till infinity. What you are what you will be carving out in r 3 will be this sort of the trace of this star sort of shape. That will not be a convex set do because it will have these this sort of this kind of boundary right.

So, this is a so, but that is necessary, but that is a cone ok. So, it is so that is that sort of set is not necessarily a cone is not is a case of a convex set that is not a cone. So, suppose if I gave you two cones C 1 C 2 if I if C 1 C 2 are cones, then what about what can you say about C 1 intersection C 2?

What about the intersection of two cones? The intersection of two cones is a cone, but can also be a trivial cone see it can happen that you are the two cones are like say here is these two lines that they intersect on these two rays like this they are individually cones, but the intersection is only the origin right.

So, the intersection can this is one of the you can have the problem that the intersection of two cones is just is this sort of trivial cone ok. Now, so we saw that the cone is not necessarily convex. So, this creates this problem of defining analogous things analogous to conic two convex combinations. So, if you if I give you points x 1, x 2 till x k that belong to say a set S ok. So, let S be in a certain R n and I give you points x 1, x 2, x k belonging to S, thishis and then the then this quantity summation alpha i xi.

Let us call this y summation alpha i xi i running from 1 to k. Where now the alphas do not there is no restriction on there sum, but they are just greater than equal to 0, all the alphas are just greater than equal to 0. This is what is called a conic combination ok. Now, if I take the set of all conic combinations that means, the set like this summation alpha i xi i running from 1 till k; alphas are greater than equal to 0 x 1 till x k belong to S and k is any natural number right.

So, this is the set of all conic combinations of points in S. This is what we call the conic hull of S or the cone generated by S. Now, can you tell me, how is this related to the smallest cone that contains S? How is this related to the smallest cone that contains S? It is so it is not the intersection of all cones that contain S, it is not the intersection of all cones that contains S and the reason for that is this kind of conic hull is always a convex set.

So, this the conic hull is a convex cone that is because of the way we have defined it you will see you, it is very easy to see actually that if I take convex combinations of two points like this; two points in the conic hull that convex combination also lies in the conic hull right. So, conic hull is the is not the smallest cone, but the smallest convex cone that contains S.

So, just as an example a very extreme example. Suppose I take the set S as just having two points x 1 x 2 in r 2. So, my set is actually these two points, these two points x 1 x 2. What is the smallest cone containing these two points? It is those two lines or to rays starting from the origin and passing one passing through x 2, one passing through x 1 right.

So, the smallest cone containing these two would have been this. So, a point here would not be in the in this the smallest cone containing x 1 x 2. So, what about the conic hull what is the conic hull of S of this x 1 x 2? So, when I take combinations alpha 1 x 1 plus alpha 2 x 2 where alpha 1 and alpha 2 are non negative. What am I generating effectively you know just think of your parallel (Refer Time: 14:38) prepared whatever is called from you know high school geometry.

So, you have x 1 x 2 alpha 1 x 1 plus alpha 2 x 2 would give you. So, this would be x 1 plus x 2 alpha 1 x 1 maybe alpha 1 would be x 1 would be here alpha 2 x 2 say here. We would this would be alpha 1 x 1 plus alpha 2 x 2 correct and by doing this when I as I range over alpha 1 alpha 2 that are non negative, I will basically span this entire region in right.

So, the conic hull is usually larger than the smallest cone, but it is the smallest convex cone that contains the cell, that you can see its self evident here. Smallest convex cone containing x 1 you know x 1 and x 2 is the v with the you know the interior (Refer Time: 15:43) alright.

So, I will gives me chance to define hyperplane. So, hyperplane is defined like this. So, you will take a vector a in R n take a scalar b in R. A hyperplane is simply all the x is that satisfy a transpose x equals b. So, hyper plane H is the sort of set is x such that a transpose x equals b you can also write it in the following way.

You can also write this as, suppose you know that x 0 is such that a transpose x 0 is equal to b. That means, it is a point on the it is a point on the hyperplane. Then this is the same as writing x such that a transpose x minus x 0 equals b. Sorry x minus x 0 equals 0 right. So, what this means is take any two points x and x 0 that lie on the hyperplane. Where does the vector a point? The point the vector a point is orthogonal to the segment x minus x 0 right.

So, in on a in a hyperplane, when you write a hyperplane in this form as x such that a transpose x equals p the vector a is the is your is the normal to the hyperplane. So, this is the normal this is the normal to the hyperplane right. So, because the hyperplane can be written in this form x such that a transpose x minus x 0 equals 0.

Then it that also means that it the hyperplane is actually nothing but x 0 plus this a perp. What is a perp? A perp is that subspace which is perpendicular to a right. So, it is the orthogonal complement of a. So, the entire subspace which is perpendicular to a. So, this is the a hyperplane is different from what is called the half space.

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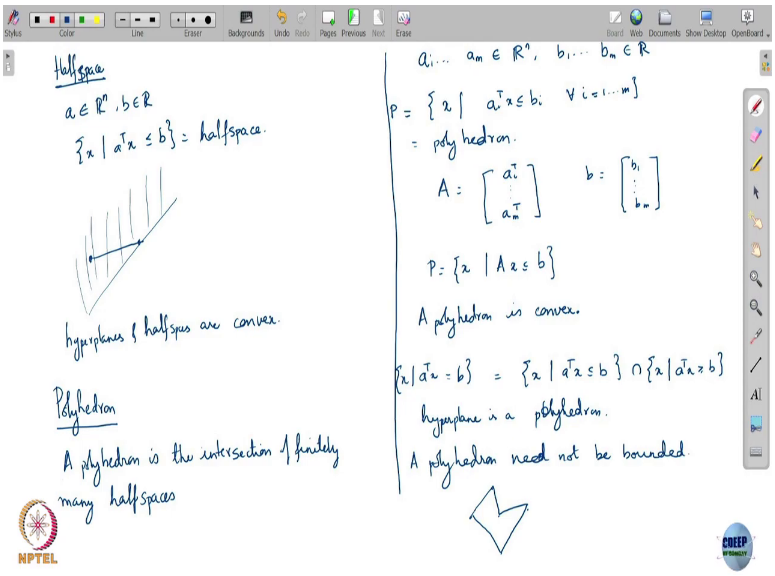

The half space is again let us suppose you have a in R n b in R and a half space is x such that a transpose x less then equal to b. So, a half space is a hyperplane and one side of it ok. This is what we defined this sort of set is what is called a half space. So, is a half plane is a hyperplane a convex set. Yes, right is a half space a convex set, is a half space a convex set? Yes, so it is this hyperplane and all the all of this right. So, I can take any two points in it and the segment joining those two points is always in the half space ok.

So, half a hyperplane and a half and half spaces hyperplanes and spaces these are always convex. Now, remember I asked you; what if I took the wall here was a affine set. But the wall and the air in the room all of that together that was not an affine set.

So, the wall is an example of that is actually a hyperplane wall and all the air on one side of it that is a half space ok. So, hyperplane in particular is always affine, but half space is not affine. Final concept for today is that of a polyhedron a polyhedron is a polyhedron is the intersection of half spaces intersection of finitely many half spaces ok.

So, if I so what that means, is I need to I need vectors like this say a 1 till a m all belonging to R n and I need b 1 till b m all scalars, then you look at something like this. The x is such that they are common to the m half spaces that are created by these parameters right.

So, x such that a i transpose x is less than equal to b i for all i from 1 to m this is a this set P is a polyhedron ok. Another way to write this is to simply take put all these a i transposes and make them rows of a matrix a stack up the b's into a column vector b, into a column vector b 1 to stack up the b i's in into a column vector b.

So then your polyhedron p can simply be written as x such that a x is less than equal to b. This for us is a polyhedron. So, is a polyhedron a convex set? Yes, it is a intersection of half spaces who are there that are themselves convex ok.

So, polyhedron is always convex, a polyhedron is always convex, is a polyhedron bounded? Is a polyhedron bounded? The answer is no, because a half space is a polyhedron, a hyperplane is a polyhedron right, these are not bounded sets. A hyper plane is a polyhedron can you explain, why a hyperplane is a polyhedron?

Student: (Refer Time: 24:47).

Yeah, I can think of it as an intersection of two half spaces one where you have. So, a hyperplane like this x such that a transpose x equals b is the intersection of x such that a transpose x less than equal to b with x such that a transpose x greater than equal to b right. So, it is an intersection of these two half spaces. So, consequently it is also a; it is also a polyhedron hyperplane is a polyhedron, every hyperplane is a polyhedron every. So, a polyhedron need not be bounded, polyhedron need not be bounded ok.

Now, polyhedron is should not be confused with another common term that is used which is polygon ok. A polygon often stands for something like this, some set like this is often what is referred to as a polygon, that is not what we mean by polyhedron. This in this sort of set is actually it this sort of set this the one I have drawn right here this is actually not even convex ok. So, a polyhedron for us is the intersection of half spaces polyhedron need not be bounded.

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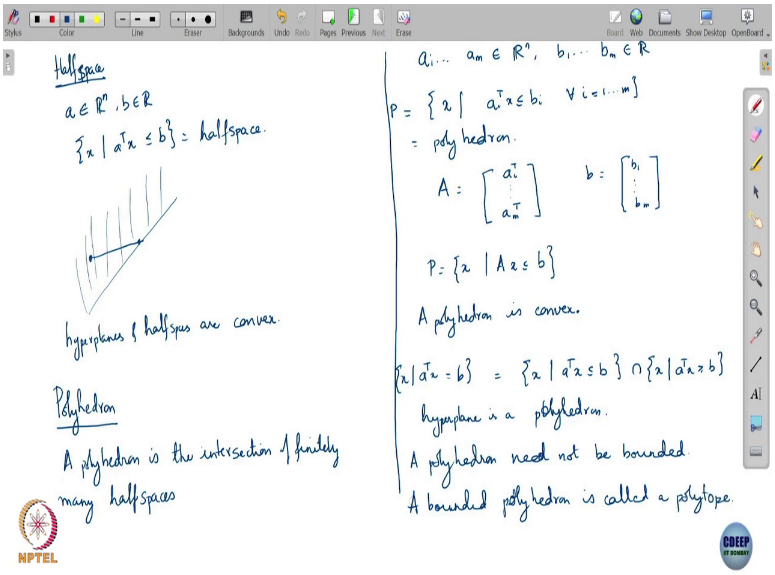

But if it is bounded so a bounded polyhedron is called a polytope, bounded polyhedron is called a polytope ok. Now what I want you to do as an exercises to think of, what a polytope would look like. It is an intersection of half spaces such that the overall intersection turns out to be bounded. Bounded means, it can be put inside some ball of finite radius. So, but it is itself an intersection of finitely many half space ok.

So, we can. So, think about it and we will discuss this next time.