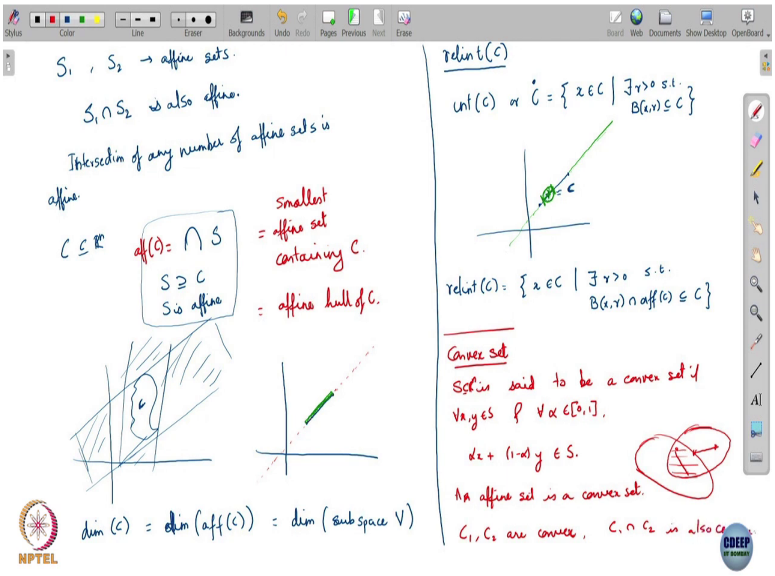
Optimization from Fundamentals
Prof. Ankur Kulkarni
Department of Systems and Control Engineering
Indian Institute of Technology, Bombay

Lecture – 8B
Convex Analysis - II

(Refer Slide Time: 00:22)



Now, what if suppose if I gave you two affine sets. Suppose, S 1 is an affine set, S 2 is an affine set; suppose these are affine sets, what can you say about the intersection? S 1 intersection S 2; what can you say about the intersection S 1 intersection S 2? Think about it; think about it this way, suppose we know this wall is an affine set now say there is consider another wall perpendicular to it.

So, the say the wall behind that is also an affine set. When these; where do these two intersect? They intersect at that corner line there ok; just imagine that is extended from in both directions, all the way till infinity. Then, now that intersection is that an affine set?

Student: (Refer Time: 01:15).

That is also an affine set right ok. So, the intersection; so if I take any two affine sets, this is also is also affine ok. So, if the intersection of two affine sets is affine; what about the intersection of; so, the intersection of any number of affine sets is actually affine.

Now, if I give you a set C; some set C in R n and then I look at the following quantity; the intersection of all sets S; such that S contain C and S is affine. So, ranging over all affine sets that contain C; if I take the intersection, what can you say about this particular object?

So, you are taking somebody like this; somebody C that lies some set C like this and then you are taking all affine sets that contain C. So, the thing that I have boxed here that is itself an affine set because it is an intersection of affine set. So, this is itself an affine set and in fact, so since it is itself an affine set and it must contain C because it all of the all the S is also contain C right. So, it is an affine set that contain C right.

So, it is in fact included in this list here with the list; when you run over all this affine sets that contain C; this intersection there is also one of them right. So, what does that mean? This intersection is actually the smallest affine set containing C right. So, let us try to imagine what this sort of thing looks like; there is a name for it let me tell you the name; this is what is called the affine hull of C ok. What is this set look like? So, suppose we were in R 2, this is the way I have drawn it here; we are in R 2 and this is my set C.

What is the give me one affine set that contains this set C. The plane R 2 itself right; that is one example of an affine set that contain C. Is there any other affine set that contain C? If you think about it, you in R 2 there is just not enough space to fit in another affine set which is not

R 2 itself. So, yeah; so let us me draw, let me draw another figure for you; just to make this easier.

Consider, suppose just this little segment here. What is the affine hull of this? It is the line that passes through these points right. So, why would this be the affine hull? Take give me an; can you tell me affine sets that contain this; this segment. R 2 itself is one of them right, then this line is also one of them; that is also an affine set right; so R 2 itself and this line itself. So, as you go about taking the intersection of all the affine sets that containing, what will be left with is just this line segment.

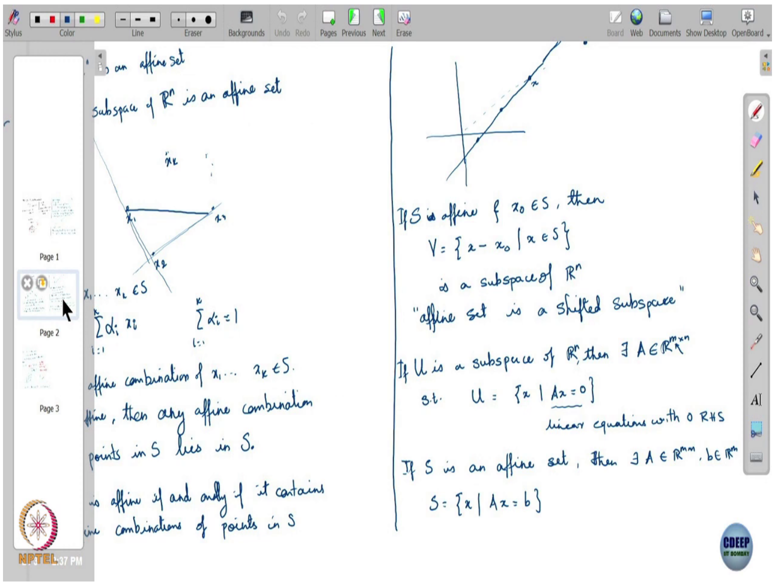
Now, suppose instead of having; instead of this being a line segment, this was not it was not a line, but say you know a pipe like this and some thickness; somes you know slight thickness ok. So, it is now actually a rectangle; now what is the affine hull? R 2 the minute it requires some thickness, basically evens; however, small you have is the only way you can contain it in an affine set is to put it into the end is to take the affine set as the full space right.

So, in this case the affine hull would become the full space. So, what is the affine hull capturing? The affine hull is essential is capturing what the essential dimension of a of that particular objective. So, you can take an object and immerse it into any high dimensional space by assigning it as many coordinates as you like. But really what is its true intrinsic dimension, you will know that only when you look at the smallest sub affine set that you need so, so that the object lies in it right.

So, the thin; the thin segment that we that I that I had first drawn, it had drawn it in R 2, but it is actually a one dimensional object. But that little square or rectangle that is the; that is actually a two dimensional object, I need an additional dimension in order to describe it right. So, the dimension of a set; we can this because of this we can now talk of the dimension of a set.

So, dimension of C is simply the dimension of the affine hull of C which is the same as the dimension of the subspace the subspace V that you would get by shifting the affine hull by shifting that affine set right; the subspace V we have previously ok.

(Refer Slide Time: 08:20)



You know this subspace V here which was by just simply shifting the space; shifting the affine set; that is the dimension of that subspace is the dimension, it can be taken as the dimension of the set C.

So, this also let us just define things like when does a set have an interior. So, this is relates to another concept called the relative interior; relative interior of a set C is defined this way. So, we remember what the interior of a set C was defined as; what was the interior of a set C? The unit of C or we denoted also by C with a circle on top.

What was this? This was all points x; x in C such that you could find a radius r around that point; such that if you took the ball of radius r around it; that ball completely lied in C right. This was the; this is the interior of a set, it is also the same as taking the union of all open sets that contain the that are contained in C. So, this was the definition of an of the interior of C.

So, by this definition if I took a little segment like this; this segment like this in R 2; does this if I take this as my set C, does this segment have a interior? No, because if I take any point here on it and take a ball around it; however, small I make the radius, the ball will always spill outside the outside the set right. So, this sort of segment does not have any interior in R 2 ok, but we can define what is called the relative interior.

So, the relative interior of C is all those points in C such that there exists an r greater than 0; such that you do you look at the ball of radius r but intersect that ball with the affine hull of C that you; so, you take the ball in the full space; then the intersected with the affine hull. That will basically slice; that will slice through the ball and what you will get is a is some you can say a low dimensional disk or something like that right; the that must lie in C.

So, this is what is called the relative interior. So, now let us look at this example. So, take this point here again; take a ball around it, this point. So, you take a ball around it; what is the affine hull of C? It is; it was this line here. So, I took the I take the ball intersected with this line; what do I, what am I left with?

A little segment right; it is so this was the this was my ball intersected with this line; I am left with just a segment like this and that segment does lie in C right. So, what the relative interior is capturing is again as I said the essential; whether the object essentially has an interior, irrespective of which what kind of space you have immersed it in or what kind of space you have used to express it. So, the ball is tied to the dimension of the ambient space which is the number of variable coordinates you have chosen right.

But it is possible that the object has an interior; if you leave us, if you leave a few coordinate ok; if you express it then only few are coordinate and a fewest number of coordinates you need is captured by the affine hull of a C ok. So, this is important because many times you can in optimization problems; when you it is possible that your optimization problem has an interior or does not have an interior.

Whether it has an interior or not depends on whether you have introduced additional variables for example. So, when you introduce additional variables; what you do is you are expressing the problem in a higher dimensional space than you need to right. So, it is possible that it in that in a difference in a larger; the higher dimensional space, it does not have an interior, but if you; but in a lower dimensional space it does and that structure can be exploited. So, this is what is called the relative interior ok.

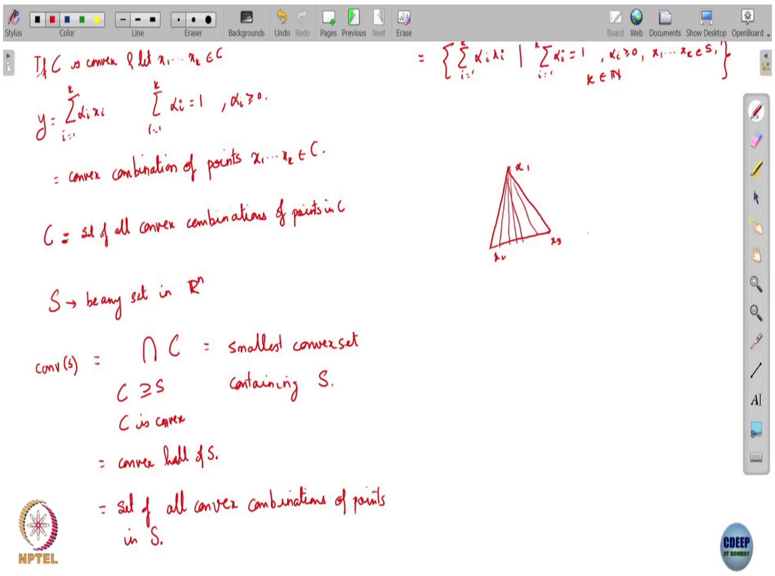
So, then the most important concept for us is; what is the concept of a convex set. So, S is said to be; so let us call a it is a sub set in R n is said to be a convex set if for all x comma y in S and for all alpha belonging to 0, 1; alpha x plus 1 minus alpha y be longs to S ok. So, I am sure you have seen this at some point of time.

So, take any two points in the set, take the line segment joining the two point; the entire line segment must; then the set is said to be convex set. So, convex set for example, looks like this; this is an example of a convex set. I two points here the entire segment join lies in it; I think these two points this entire segment lies in it and so on. So, let me ask you a few simple things; is a convex, is an affine set or convex set?

An affine set is always a convex set because of course; it contains not only the segment, but the entire line joining the passing through the two points right. So, an affine set is a convex set; is it possible for a convex set to be say open for example? No, there is no problem at all an convex set can be open, convex set can be closed openness and closeness has nothing to do with whether it is convex or not; these are completely separate properties ok.

What if I; what if I take the intersection of two convex sets? So, suppose C 1, C 2 are convex; then what about C 1 intersection C 2? Always convex right; here is a convex set for example; I intersected with another convex set the intersection is also convex ok.

(Refer Slide Time: 16:20)



So, if I; if C is convex and I take and let x 1 till x k be points in C ok. Then this thing here again, let us call this y again; so alpha i; x i ranging from 1 to k; k were now alphas sum to 1 as before. So, this if I had left it as summing to 1, then this would be an affine combination of points x 1 till x k ok.

So, now, in this case I am going; they are going to sum to 1 and in addition to that I am also going to impose that alpha is are all greater than equal to 0 ok. So, these are non negative and sum to 1 and then in this case this is called a convex combination. So, if C is convex; then what can you say about convex combinations of points in C? So, you take some k points; take any convex combination of those k point; what can you say about the convex combination y?

Student: (Refer Time: 18:01) C.

Also lie in C right. So, you can imagine; it is not; so C is, so it is not very hard to imagine, I will let me just show you. So, take two points here; say let us take three points for example, the convex combination of these two points; any convex combination which will make the segment that lies in C, this segment lies in C, this segment lies in C; all three of those segments lie in C.

Now, I can do more I; now that I know the center segment lies in C, I can take this sort of thing and take this sort of thing; eventually start taking any of these kind of segments and I would have covered the entire triangle right.

So, the every possible convex combination of the three vertices of this triangle; would be covered by this right. So, this is not very hard to see. What is more interesting is actually the following. So, C is actually the same as; C is equal to the set of all convex combinations of points in C. So, if you take a all possible convex combinations of points in C; of course, they lie in C, but C does not contain any additional points on top of that. So, C is the set of all convex combination of points in C ok.

So, now since arbitrary intersections of convex sets are convex; we can again do what we did with an affine with an affine hull. So, let us; so, let us S be any set; let S be any set in R n. And now and define this thing which is the intersection of all sets C that contain S and C is convex. So, this is the intersection of all convex sets that contain S right; this is, so what sort of set is this?

This is itself a convex set; moreover it is a convex set that contains S. So, it is also the smallest convex set that contains S right. So, this is actually equal to; this is the smallest convex set containing; containing S. There is a name for this; this is what is called the convex hull of S. Now, can you describe for me the convex hull of a set in a different way? Can you describe for me the convex hull of a set in a different way? One way is to take the intersection of all convex sets that contain that set ok.

So, you put together all the convex sets, keep taking the common whatever is the area that is common in it the; that will give you a smallest convex set that contains the set that is; that is one way of defining the convex hull. In S right; so that is, so that is correct. So, this is also the set of all convex combinations of points in S.

So, you which is; which another way of saying the same thing is this is the summation x i; alpha i, i ranging from 1; write this on the other side. The set of vectors like this; summation alpha i; x i, i ranging from 1 till k ok; now, my alphas are; I need to make sure this is a convex combination. So, my alpha sum to 1; i ranging from 1 to k; this sums to 1, my alphas are also nonnegative, this is; this ensures that I have a convex combination, does this; is this it?

This is defined for me; yeah I need to also allow for choosing any exercise that belong to belong to the set S right. So, i; so x 1, till x k; all belong to S. What else? Is that it? So, without the without saying that x 1 till x k, if I belong to S.

So, if I removed this; if I remove the constrained, this requirement at x 1 till x k belongs to S, then I will be; I will be looking at the all convex combinations of specific k point; x 1 till x k right, that is not what we are looking for. We want all possible convex combinations of any number of any points.

So, I need to allow for x 1 to x k to also vary, but is that all? If I stop at this, this would be in convex combinations of just k point; it will not include k plus 1 point; convex combinations of k plus 1, k plus 3 etcetera right; I also need to let k be N; yes yeah they could be different. So, there is another result actually is called Caratheodory’s theorem which could gives you a bound on how many k do you need ok.

So, that we do not have time to discuss that with; it turns out you do not need to take too many more than the dimension of the set. So, that is a separate result all together, but a priori without any without that result, I cannot say anything about this. See, for is a one way to check this is to say if k is 2 for example, I will get convex individual segments right.

So, just in the case; let us suppose if I took three points x 1, x 2, x 3; I will get three segments like this ok. But now if I took, but then you are; if you allow the excess to very then I can start generating of some additional segments. So, it turns out; so in since we are in two dimensions here 2 is enough ok. You do not need to take 3 at a time, but then this; this is the this is yeah, it is much more complicated than that I would takes an; it is a complete; it requires a separate result all together.